

Volume of Solid of Revolution

- i. Draw the enclosed region in the xy -axes, and clearly mark the endpoints.
 - ii. Draw a typical reference rectangle either **perpendicular** or **parallel** to the axis of revolution.
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How to choose a method:

Use Disk Method when

- The reference rectangle is **perpendicular** to the axis of revolution, **and**
- The axis of revolution is **completely attached** to the enclosed region.

Use Washer Method when

- The reference rectangle is **perpendicular** to the axis of revolution, **but**
- The axis of revolution is **not completely attached** to the enclosed region.

Use Cylindrical Shell Method when

The reference rectangle is **parallel** to the axis of revolution.

Horizontal Axis of Revolution $y = b$

Disk Method:

$$V = \pi \int_{x_1}^{x_2} L^2 dx,$$

L is the length of the reference rectangle.

Washer Method:

$$V = \pi \int_{x_1}^{x_2} [R^2 - r^2] dx,$$

R is the radius of the outer circle and r is the radius of the inner circle.

Shell Method:

$$V = 2\pi \int_{y_1}^{y_2} DL dy,$$

L is the length of the reference rectangle, and

$D = b - y$ when the axis of revolution is above the enclosed region.

$D = y - b$ when the axis of revolution is below the enclosed region.

Vertical Axis of Revolution $x = a$

Disk Method:

$$V = \pi \int_{y_1}^{y_2} L^2 dy,$$

L is the length of the reference rectangle.

Washer Method:

$$V = \pi \int_{y_1}^{y_2} [R^2 - r^2] dy,$$

R is the radius of the outer circle and r is the radius of the inner circle.

Shell Method:

$$V = 2\pi \int_{x_1}^{x_2} DL dx,$$

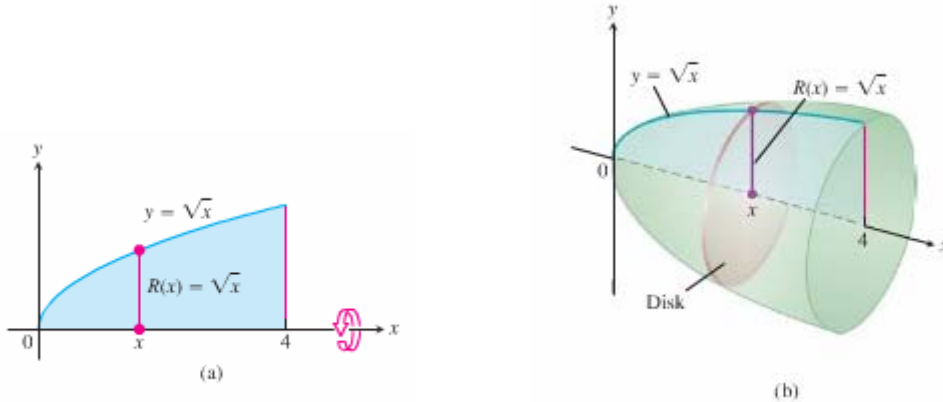
L is the length of the reference rectangle, and

$D = a - x$ when the axis of revolution is right of the enclosed region.

$D = x - a$ when the axis of revolution is left of the enclosed region.

Example: Disk Method

The region between the curve $y = \sqrt{x}$, and the x -axis for $0 \leq x \leq 4$ is revolved about the x -axis to generate a solid. Find its volume.



Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

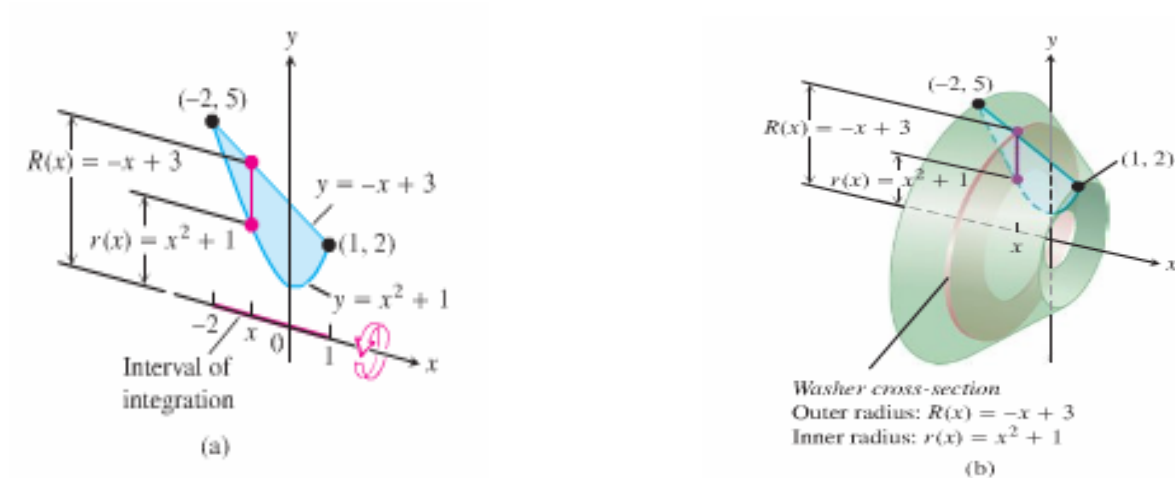
$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2 dx \\ &= \int_0^4 \pi[\sqrt{x}]^2 dx && R(x) = \sqrt{x} \\ &= \pi \int_0^4 x dx = \pi \left. \frac{x^2}{2} \right|_0^4 = \pi \frac{(4)^2}{2} = 8\pi. \end{aligned}$$

Now you try:

Find the volume of the solid generated by revolving the region between the y -axis, the curve $xy = 2$, and $1 \leq y \leq 4$ about the y -axis.

Example: Washer Method

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x -axis to generate a solid. Find the volume of the solid.



Evaluate the volume integral.

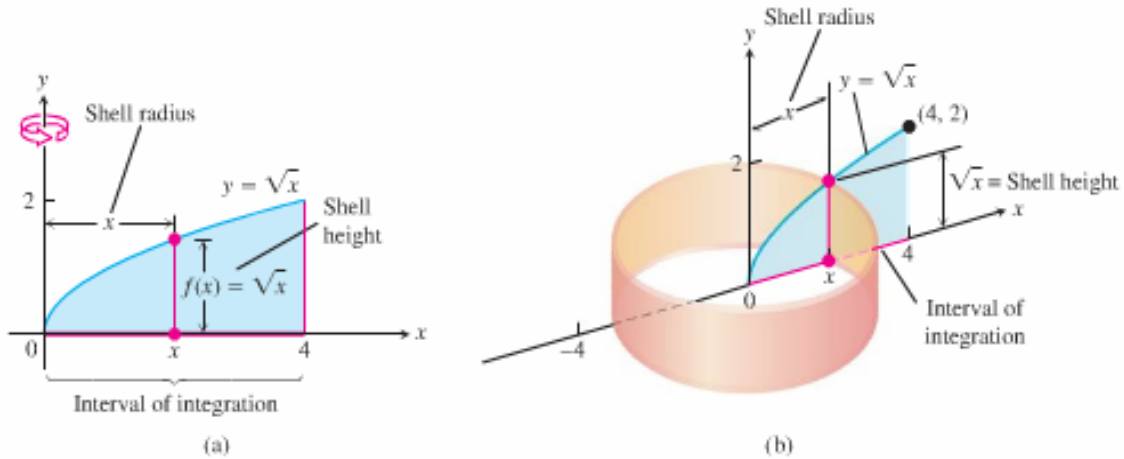
$$\begin{aligned}
 V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx \\
 &= \int_{-2}^1 \pi((-x + 3)^2 - (x^2 + 1)^2) dx \\
 &= \pi \int_{-2}^1 (8 - 6x - x^2 - x^4) dx \\
 &= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^1 = \frac{117\pi}{5}
 \end{aligned}$$

Now you try:

The region bounded by $y = x^2$, and $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find its volume.

Example: Cylindrical Shell Method

The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the y -axis to generate a solid. Find the volume of the solid.



Distance between the Reference Rectangle and the Axis of Revolution

Length of the Reference Rectangle

$$\begin{aligned}
 V &= \int_a^b 2\pi \left(\text{shell radius} \right) \left(\text{shell height} \right) dx \\
 &= \int_0^4 2\pi(x)(\sqrt{x}) dx \\
 &= 2\pi \int_0^4 x^{3/2} dx = 2\pi \left[\frac{2}{5}x^{5/2} \right]_0^4 = \frac{128\pi}{5}.
 \end{aligned}$$

Now you try:

The region bounded by $y = x^2$, $y = x^2 + 3$, and $1 \leq x \leq 3$ is revolved about the $x = -1$ to generate a solid. Find its volume.