Volume of Solid of Revolution

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i. Draw the enclosed region in the xy – axes, and clearly mark the endpoints.
ii. Draw a typical reference rectangle either **perpendicular** or **parallel** to the axis of revolution.

How to choose a method:

Use Disk Method when

- The reference rectangle is **perpendicular** to the axis of revolution, **and**
- The axis of revolution is **completely attached** to the enclosed region.

Use Washer Method when

- The reference rectangle is **perpendicular** to the axis of revolution, **but**
- The axis of revolution is **not completely attached** to the enclosed region.

Use Cylindrical Shell Method when

The reference rectangle is **parallel** to the axis of revolution.

Horizontal Axis of Revolution $y = b$

Disk Method:

$$V = \pi \int_{x_1}^{x_2} L^2 dx,$$

L **is the length of the reference rectangle.**

Washer Method:

$$V = \pi \int_{x_1}^{x_2} \left[R^2 - r^2 \right] dx,$$

R **is the radius of the outer circle and** r **is the radius of the inner circle.**

Shell Method:

$$V = 2\pi \int_{y_1}^{y_2} DL \; dy,$$

L **is the length of the reference rectangle, and**

$D = b - y$ **when the axis of revolution is <u>above</u> the enclosed region.**

$D = y - b$ **when the axis of revolution is <u>below</u> the enclosed region.**

Vertical Axis of Revolution $x = a$

Disk Method:

$$V = \pi \int_{y_1}^{y_2} L^2 dy,$$

L **is the length of the reference rectangle.**

Washer Method:

$$V = \pi \int_{y_1}^{y_2} \left[R^2 - r^2 \right] dy,$$

R **is the radius of the outer circle and** r **is the radius of the inner circle.**

Shell Method:

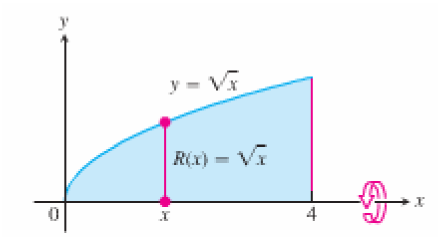
$$V = 2\pi \int_{x_1}^{x_2} DL \; dx,$$

L **is the length of the reference rectangle, and**

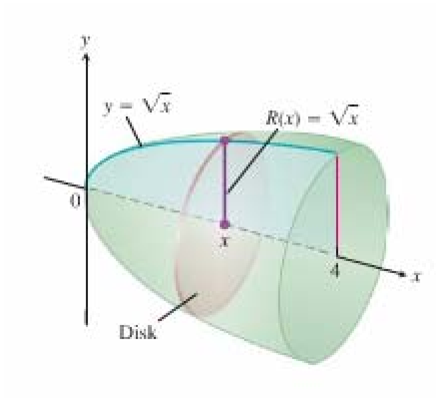
$D = a - x$ **when the axis of revolution is <u>right</u> of the enclosed region.**
$D = x - a$ **when the axis of revolution is <u>left</u> of the enclosed region.**

Example: Disk Method

The region between the curve $y = \sqrt{x}$, and the $x-$ axis for $0 \le x \le 4$ is revolved about the $x-$ axis to generate a solid. Find its volume.



(a)



(b)

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

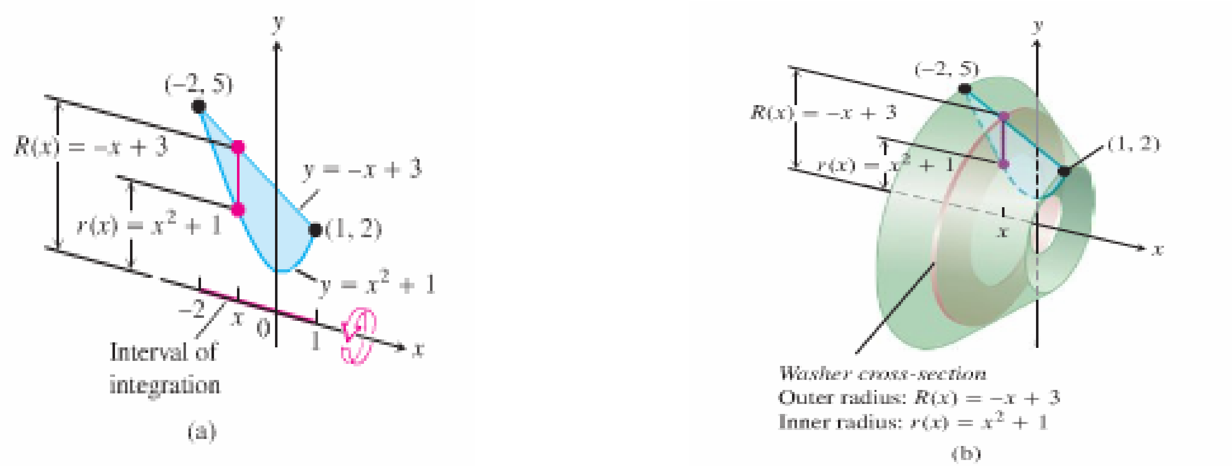
$$
\begin{aligned}
V &= \int_a^b \pi[R(x)]^2\,dx \\
&= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \qquad R(x) = \sqrt{x} \\
&= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\bigg]_0^4 = \pi\frac{(4)^2}{2} = 8\pi.
\end{aligned}
$$

Now you try:

Find the volume of the solid generated by revolving the region between the $y-$ axis, the curve $xy = 2$, and $1 \le y \le 4$ about the $y-$ axis.

Example: Washer Method

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x– axis to generate a solid. Find the volume of the solid.



Evaluate the volume integral.

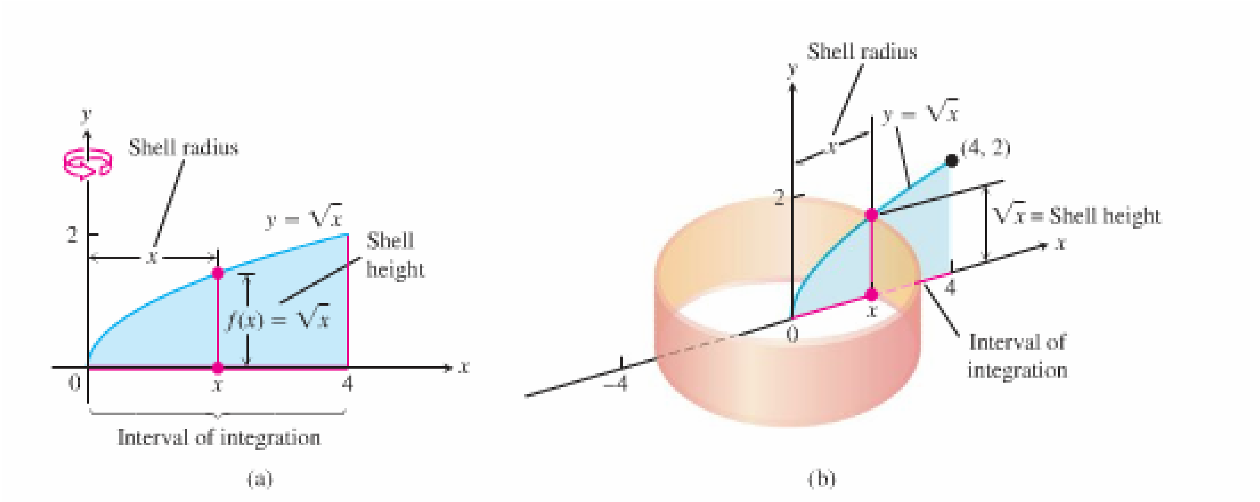
$$
\begin{aligned}
V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx \\
&= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\,dx \\
&= \pi \int_{-2}^{1} (8 - 6x - x^2 - x^4)\,dx \\
&= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}
\end{aligned}
$$

Now you try:

The region bounded by $y = x^2$, and $y = 2x$ in the first quadrant is revolved about the y– axis to generate a solid. Find its volume.

Example: Cylindrical Shell Method

The region bounded by the curve $y = \sqrt{x}$, the x– axis, and the line $x = 4$ is revolved about the y– axis to generate a solid. Find the volume of the solid.



Distance between the Reference Rectangle and the Axis of Revolution

Length of the Reference Rectangle

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx$$

$$= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi \left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}.$$

Now you try:

The region bounded by $y = x^2$, $y = x^2 + 3$, and $1 \le x \le 3$ is revolved about the $x = -1$ to generate a solid. Find its volume.