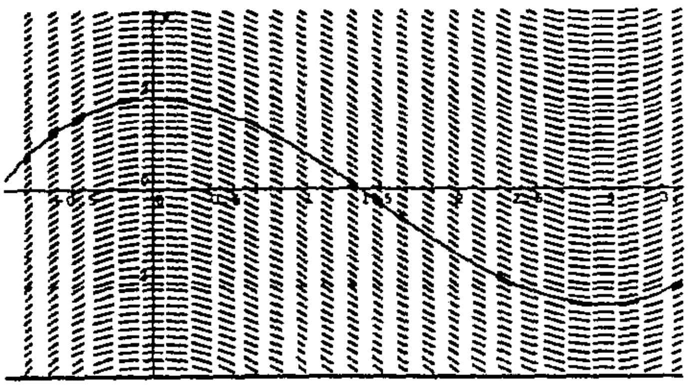
VISUALIZING DIFFERENTIAL EQUATIONS: SLOPE FIELDS & EULER'S METHOD



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Slope Fields & Euler's Method

Slope Fields and Euler's Method are two of the "new" topics on the BC Syllabus. As such they have caused some concern. Where, all of a sudden, did they come from? Why do we need them? When should we include them? If I only teach AB do I have to be concerned about them?

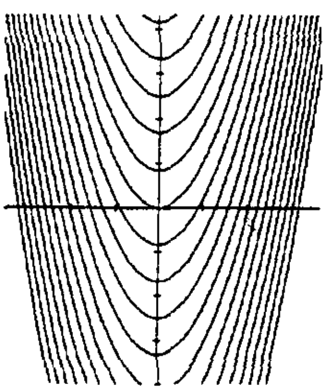
In a few words: A slope field is the graphical representation of a differential equation. Euler's method is a numerical method for approximating the solution of a differential equation. You've probably been teaching Euler's method for years under a different name.

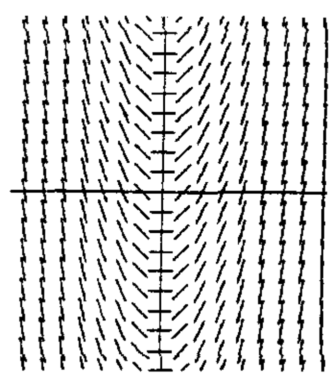
Slope Fields

Many real world phenomena are modeled by differential equations. The mathematical use of a slope field is to approximate the graph of a solution to a differential equation. Their pedagogical use is to help students better understand the solution of differential equations. Multiple representations of all the major ideas in mathematics is, as it should be, the overriding strategy of mathematics teaching today. Writing the equations is one thing, but seeing the solutions by plotting slope fields removes the abstractness from the symbolic representation. Also many differential equations cannot be solved analytically, yet the graph of the solution can be easily approximated by using slope fields and Euler's Method.

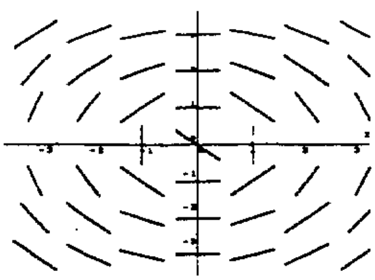
What is a slope field? It is a graph of short line segments whose slope is determined by evaluating the derivative at the midpoint of the segment. But it's really better to see one....

Teaching Idea: Prepare a transparency with a family of functions such as the one shown in the figure at the left. The equation is $y = x^2 + B$ for various values of B. Use any function you want, of course. Put the transparency on an overhead projector and cover it with another transparency. With a marker pen, moving across the page short, draw short segments tangent to the curves. Go down a line and draw another line of segments. Continue down the page. When you are done, remove the first transparency (the one with the curves) and the remaining one, which should look like the figure on the left, is the slope field. This is the strictly graphical introduction to the slope field.





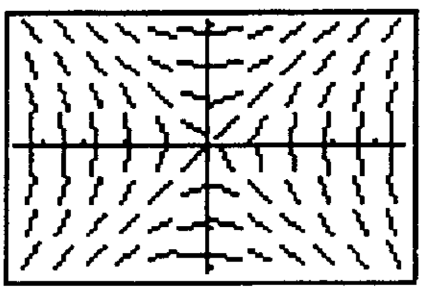
Teaching Idea: This is the numerical introduction. Put a differential equation on the board, perhaps $\frac{dy}{dx} = -\frac{x}{y}$. Give each member of the class one or two points — the grid points of the



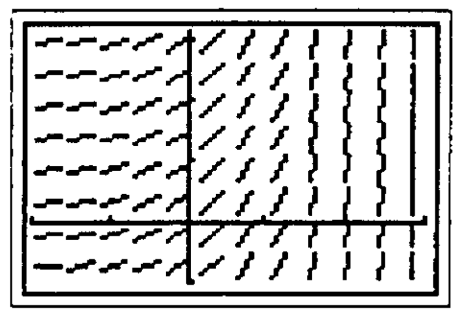
square whose diagonal runs from (-3, -3) to (3,3). Each student calculates the value of *dy/dx* at his or her points. Then, again on the overhead projector, have each student graph their segments on a coordinate grid, through the point they used to calculate their slope. This should give a slope field similar to the one on the left. The circle can be seen lurking in the slope field.

Teaching Idea: On page 12 there are programs for genreating slope fields on the TI-82, the TI-83 and the TI-85. The TI-86 has a built in slope equation in the field utility. These may be used to produce the slope field of a function. For each, enter the differential equation as the Y1 equation in the editor and run the program. The slope fields are shown below were produced on a TI-83:

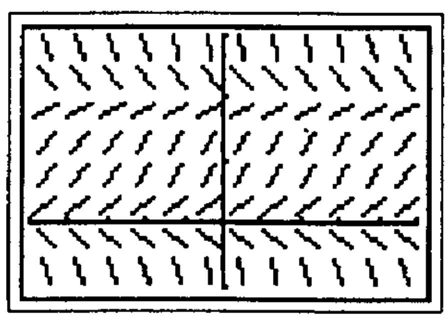
$$\frac{dy}{dx} = \frac{x}{y}$$



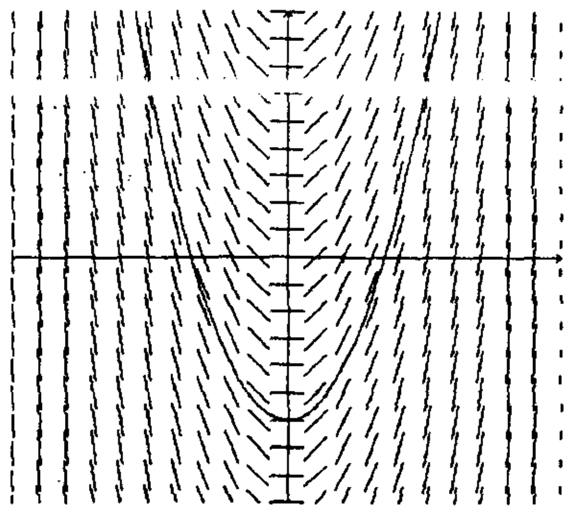
$$\frac{dy}{dx} = e^x$$



$$\frac{dy}{dx} = y(2-y)$$



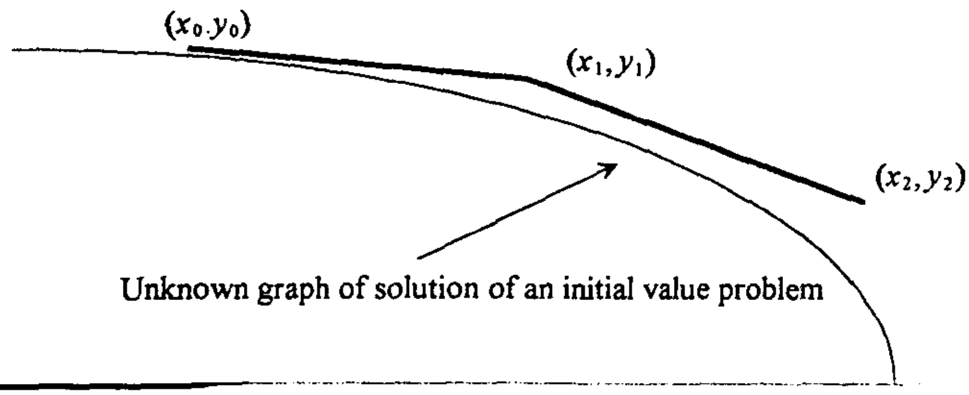
The solution curves are hiding in the slope field. Given one point of the particular solution curve, you can sketch the graph from that point, in both directions, to see the graph of the solution. The initial value problem $dy/dx = 2x$ with $y = -6$ when $x = 0$ is shown on the next page. Notice how the graph follows the flow of the slope field starting at (0, -6) and going both left and right. Another example is shown on the cover. (The equation is $dy/dx = x(x-3)$ with the initial point (0,2).)



Some *exercises* with slope fields are included at the end of this paper. We now consider Euler's Method.

Euler's Method

By the time you are ready to introduce Slope Fields and Euler's Method students have a good understanding of the idea of *local linearity*: differentiable functions when looked at "close up" appear to be straight lines. This means that over small distances the tangent line to a graph is close to the graph and therefore may be used to approximate the value of the function near the point of tangency. Euler's Method: Given point on the solution graph of a function whose first derivative is known, a short tangent segment, with one end at the given point, is drawn or calculated. Its other end is used to approximate the value of the function with the same x-coordinate. This point, (x_1, y_1) and the value of the derivative *there* (not on the actual graph), is used to calculate another point (x_2, y_2). Continuing the process produces an approximation of the graph or a table of approximate values for the function.



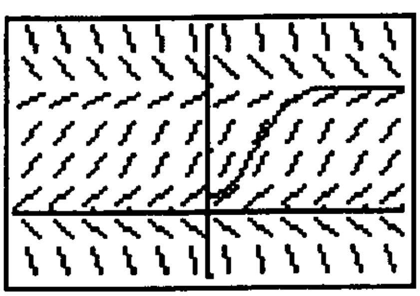
The y-coordinate of the point at the right end of the tangent line above is the value calculated by Euler's Method. In the simple case of $y = f(x)$, you may recognize it as the so-called "tangent line approximation" or the approximation by differentials. In general, the derivative may be a function of x or y. This point is then be used to calculate the next point and the next point and so on for as many points as you like. The equations for the iteration are

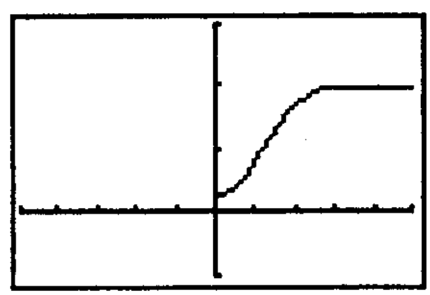
$$x_{n+1} = x_n + \Delta x$$
$$y_{n+1} = y_n + f'(x_n, y_n)\Delta x$$

The iteration may be done by hand or with a simple calculator program such as Euler Table on page 14. The derivative is entered as Y1 and the program prompts the user for the initial point and the Δx value (the step size). The program calculates the next value. Then that value is used to calculate the next and so on.

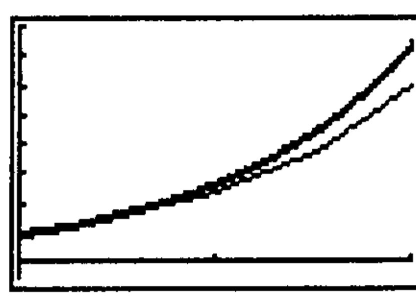
The farther one goes from the initial point (either by using large values of Δx or by doing many iterations) the less accurate are the values.

Euler's method may also be used to approximate the graph of the solution of the differential equation. Starting with an initial condition, the program Euler Graph draws the segments between the (approximate) points on the graph. This may be drawn with or without the slope field. The figures below show the solution to the initial value problem $\frac{dy}{dx} = y(2-y)$ starting with $x = 0$ and $y = 0.2$ using a step of (+0.2), shown with and without the slope field. The left side of the graph could be drawn by running the program again with the same initial point and a step of (-0.2)





Teaching Idea: Using a simple differential equation (*i.e.* one you can solve) graph the solution on your calculator and use the Euler Graph program to plot the approximate solution with it. Then you can see how close the approximation comes. The heavy graph at the left shows $y = e^x$, the solution of $dy/dx = e^x$, with the initial point (0,1). The lighter one is the Euler Graph with a step size of 0.5. (The window is [0, 2] by [-0.5, 8])



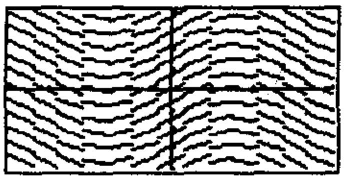
Slope Field and Euler's Method Examples

Here are some examples of problems about Slope Fields and Euler's Method. Look at them as *ways to ask questions* on these topics as well.

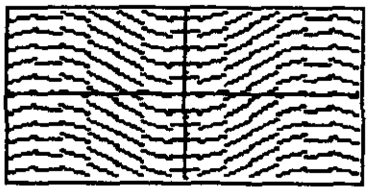
These first example is a fairly simple way to ask students about slope fields: Given the differential equation, identify its slope field. The reversal question: given a slope field, identify the differential equation is another way to ask the question.

1. Which choice represents the slope field for $\frac{dy}{dx} = \sin x$? Note: All graphs are for $-4 < x < 4$ and $-4 < y < 4$

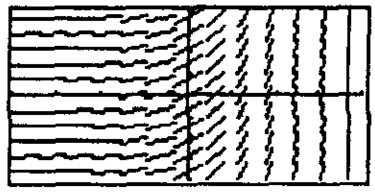
(A)



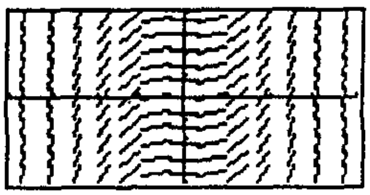
(B)



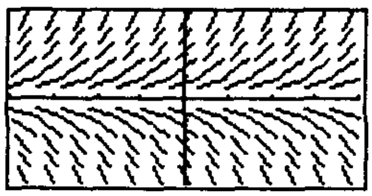
(C)



(D)

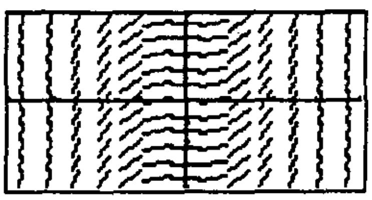


(E)



The next three problems ask the student to discuss the function whose slope field is given. Here they need to draw or picture the solution curves lurking in the slope field.

2. The slope field for differential equation $\frac{dy}{dx} = f(x)$ is shown below for $-4 < x < 4$ and $-4 < y < 4$. Which statement is true for all possible solutions of the differential equation?



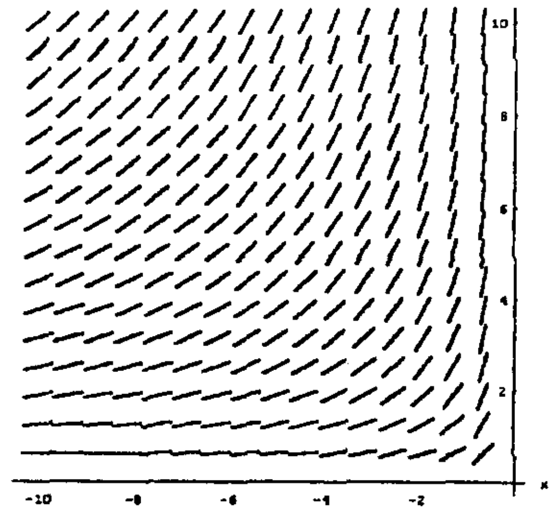
I. For $x < 0$ all solution functions are decreasing.

II. All solution functions levels off near the y-axis.

III. For $x > 0$ all solution functions are increasing.

(A) I only (B) II only (C) III only (D) II and III only (E) I, II and III

3. The figure at the left shows the part of a slope field for a differential equation in the second quadrant. Based on the figure which statement appears to be true?



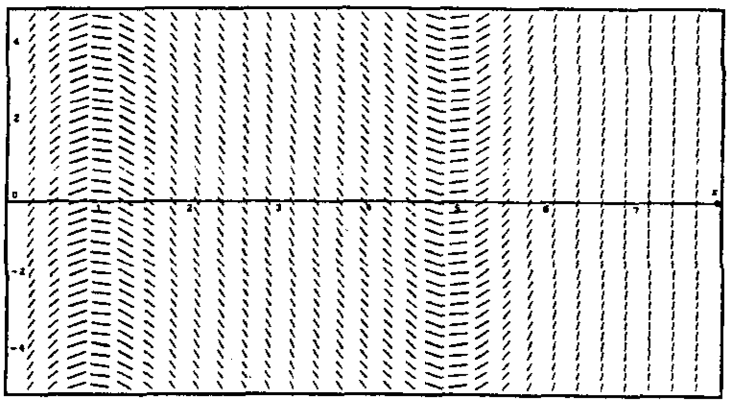
I. As x approaches zero from the left, y increases without bound

II. As x decreases without bound, y decreases without bound.

III. For points in the second quadrant $\frac{dy}{dx} \geq 0$

(A) I only (B) II only (C) III only (D) I and II only (E) I and III only

4. The figure below shows the slope field for a family of functions. Based on the figure what statement appears to be true?



I. The functions have a maximum near the point where $x = 1$

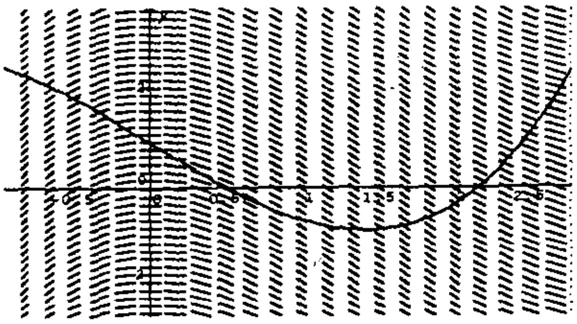
II. The functions have a maximum near the point where $x = 5$

III The functions decrease in the interval $1 < x < 5$

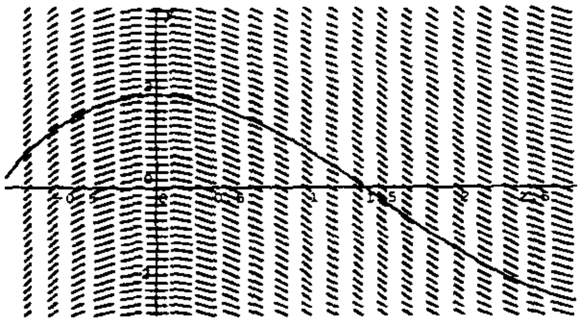
(A) I only (B) II only (C) III only (D) I and III only (E) I, II and III

5. The figures below show the slope field of the same differential equation $\frac{dy}{dx} = f'(x)$.
If $f(0) = 2$ which shows the graph of the solution of the differential equation?

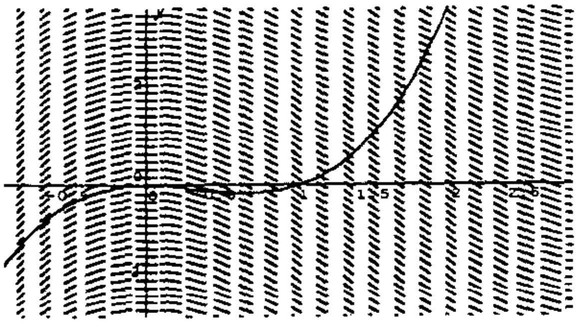
(A)



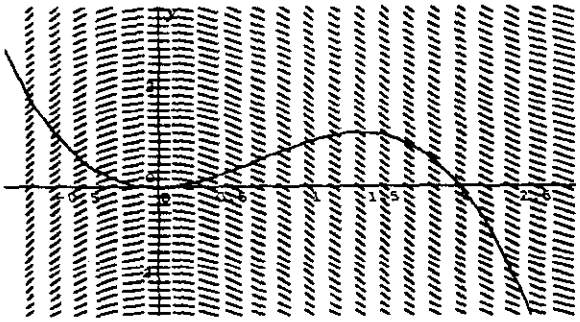
(B)



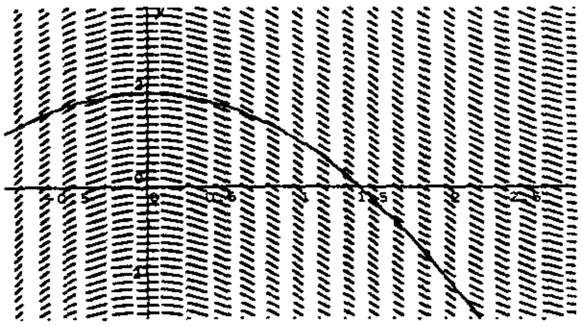
(C)



(D)

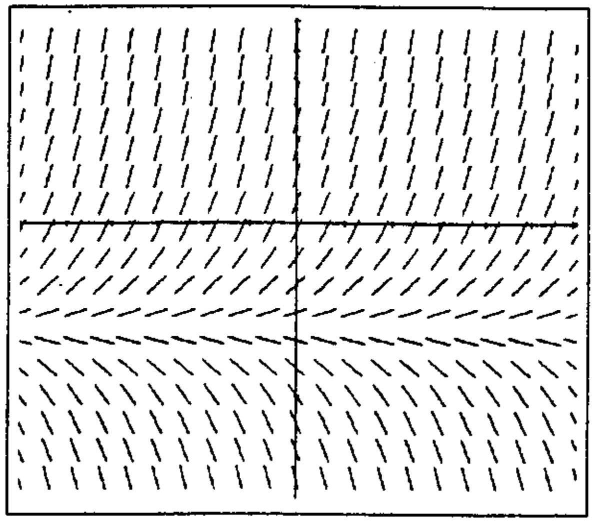


(E)



Here is are two longer problems in which, after solving the initial value problems, the student is asked a general question about the family of solutions.

6. Consider the differential equation $\frac{dy}{dx} = 6 + y$ defined for all real numbers x. The slope field for this equation is shown below in the window.



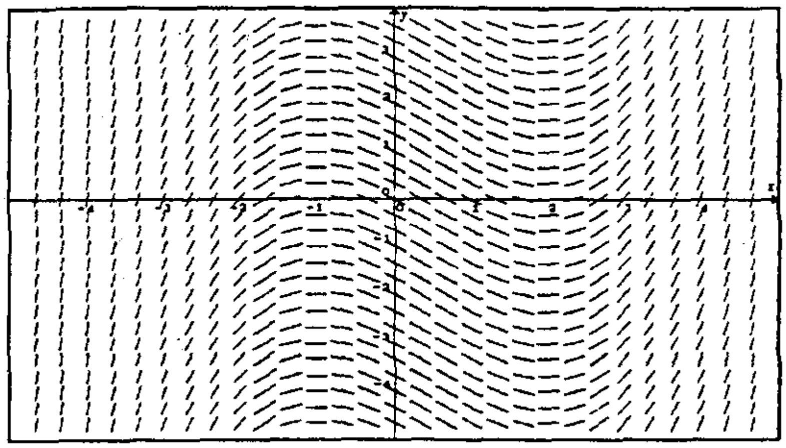
(a) Find the general solution of the differential equation in terms of an arbitrary constant C.

(b) Find the particular solution of the differential equation which meets the initial condition that $y = 0$ when $x = 0$. Sketch this solution on the slope field.

(c) Find the particular solution of the differential equation which meets the initial condition that $y = -8$ when $x = 0$. Sketch this solution on the slope field.

(d) The slope field indicates that for some of the solutions $\lim_{x \to \infty} y = +\infty$ and for other solutions $\lim_{x \to \infty} y = -\infty$. Determine the values of C for which $\lim_{x \to \infty} y = +\infty$. Show your reasoning.

7. The figure below shows the slope field for a differential equation $\frac{dy}{dx} = f(x)$. Let $g(x) = \int f(x)dx + C$ be the family of functions which are solutions of the differential equation.



(a) Determine to the nearest integer the value of x for which all of the members of the family of $g(x)$ will have a relative minimum value. Justify your answer.

(b) Determine to the nearest integer the value of x for which all of the members of the family of $g(x)$ will have a relative maximum value. Justify your answer.

(c) On the figure above sketch the member of the family of $g(x)$ for which $g(0) = -2$.

(d) For the function sketched in part (c), determine the solution(s) of $g(x) = 0$ to the nearest integer.

Two short Euler's Method problems.

8. If $\frac{dy}{dx} = xy - y^2$ and $y(1) = 3$ then $y(2) \approx$

(A) -3 (B) -1 (C) 0 (D) 3 (E) 9

9. If $\frac{dy}{dx} = \frac{y+x}{x}$ and $x = 4$ when $y = 2$, Then when $y = 2.5$ the value of x is approximately

(A) 3.25 (B) 3.50 (C) 3.67 (D) 4.33 (E) 4.75

A question on the accuracy of the local linear approximation (a.k.a. Euler's Method)

10. The local linear approximation of a function f will always be greater than or equal to the function's value if, for all x in an interval containing the point of tangency,

(A) $f'' < 0$ (B) $f'' > 0$ (C) $f' < 0$ (D) $f' > 0$ (E) $f' = f'' = 0$

Hidden in this question is an understanding of accuracy of the method.

10.

x	1.0	1.2	1.4	1.6
f(x)	2.72	A	B	12.68
f '(x)	8.20	12.80	19.30	28.50

A calculus student was preparing the table of values for f and f' shown above. Her calculator batteries died before she could calculate A and B. To finish she decided to approximate the missing values using Euler's method. Which of the choices is the ***best*** approximation using Euler's method?

(A) A = 4.36, B = 6.98

(B) A = 4.36, B = 6.92

(C) A = 3.02, B = 6.98

(D) A = 6.04, B = 9.36

(E) A = 4.78, B = 7.95

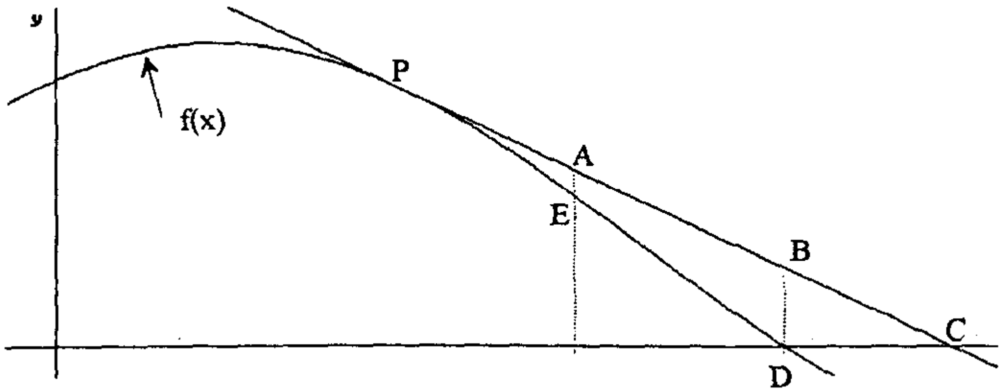
This question has three parts, but only one answer!

11. Let $f(t) = (t-3)^{\frac{1}{4}}$ and $g(x) = \int_4^x f(t)dt$

(a) Set up and evaluate a definite integral which gives the area between the graph of g(x) and the x-axis on the interval [4, 4.2]

(b) Find g(4) and g '(4) and use them to approximate g(4.2) using Euler's method with $\Delta x = 0.2$.

(c) Find $g(x)$ with no integral sign in terms of x and use it to find $g(4.2)$



12. In the figure above $\overleftrightarrow{PC}$ is tangent to the graph of $y = f(x)$ at point P. Points A and B are on $\overleftrightarrow{PC}$ and points D and E are on the graph of $y = f(x)$. Which statement is true?

I. Euler's method uses the y-coordinate of point A to approximate the y-coordinate of point E.

II. Newton's method uses the x-coordinate of point C to approximate the root of the equation $f(x) = 0$ at D.

III. Euler's method uses the x-coordinate of B to approximate a root of the equation $f(x) = 0$ at point D

(A) I only (B) II only (C) III only (D) I and II only (E) I and III only