2.1 RATES OF CHANGE AND LIMITS

Limits

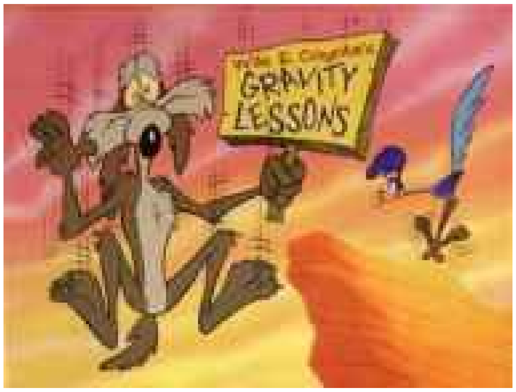
Limits are what separate Calculus from pre – calculus. Using a limit is also the foundational principle behind the two most important concepts in calculus, derivatives and integrals. Limits can be found using substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

Average and Instantaneous Speed

In pre – calculus courses, you used the formula $d = rt$ to determine the speed of an object. What you found was the object's average speed. A moving body's **average speed** during an interval of time is found by dividing the total distance covered by the elapsed time.

If an object is dropped from an initial height of h_0, we can use the position function $s(t) = -16t^2 + h_0$ to model the height, s, (in feet) of an object that has fallen for t seconds.

Example: Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.



a) Find Wile E.'s average speed for the first 3 seconds.

b) Find Wile E.'s average speed between $t = 2$ and $t = 3$ seconds.

c) Find Wile E.'s speed at the instant $t = 3$ seconds.

The problem with the part c is that we are trying to find the *instantaneous velocity*. Without the concept of a limit, we could not find the answer to part c. Using a limit to solve this problem involves studying what happens as we get "close" to 3 seconds.

Example: Find the average speed between $t = 2.5$ and $t = 3$ seconds.

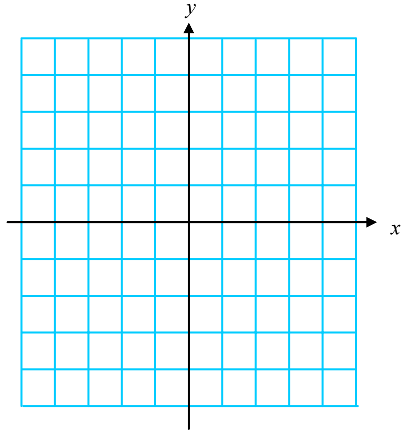
Example: Find the average speed between $t = 2.9$ and $t = 3$ seconds.

Example: Find the average speed between $t = 2.99$ and $t = 3$ seconds.

Example: Find the average speed between $t = 2.999$ and $t = 3$ seconds.

So, even though we cannot find the average velocity at exactly $t = 3$ seconds, we can discover what Wile E.'s speed is approaching at $t = 3$ seconds.

Example: Sketch the graph of $f(x)=\frac{x^2-4}{x-2}; \quad x \neq 2$.



a) What happens at $x = 2$?

b) Complete the table of values below to determine what happens as x gets "close" to 2.

x approaches 2 from the **left** ⟶ | ⟵ x approaches 2 from the **right**

x	1.5	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25	2.5
$f(x)$											

Informal Definition of a Limit

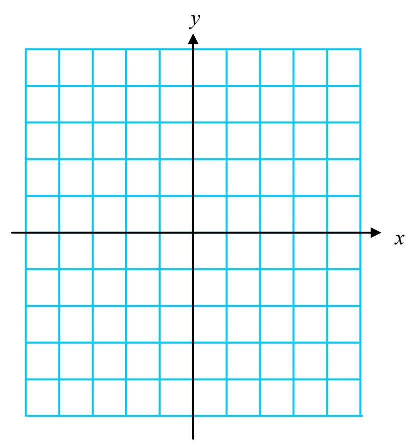
Suppose a function f is defined on an interval around $x = c$, but possibly not at the point $x = c$ itself. Suppose that as x becomes sufficiently close to c, $f(x)$ becomes as close to a single number L as we please.
We then say that the **limit of $f(x)$ as x approaches c is L**, and we write

$$\lim_{x \to c} f(x) = L.$$

c) Apply this definition to the function from above to find the $\lim_{x \to 2} f(x)$.

Example: Use the graph to find $\lim_{x \to 2} g(x)$, where g is defined as

$$g(x)=\begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$



When we say "$f(x)$ becomes as close to L as we please" in the informal definition, we mean that we can specify a maximum distance between $f(x)$ and L. This distance is given by

$$|f(x) - L| = \text{Distance between } f(x) \text{ and } L.$$

We use the Greek letter ε (epsilon) to stand for the maximum distance, so we require

$$|f(x) - L| < \varepsilon .$$

Similarly, we interpret "x becomes sufficiently close to c" to mean

$$|x - c| < \delta ,$$

where the Greek letter δ (delta) tells us how close x must be to c. Then

$$\lim_{x \to c} f(x) = L$$

means that we can make the distance $|f(x) - L|$ between the function values and L as small as we like (less than any number $\varepsilon > 0$) by making the distance $|x - c|$ between x and c sufficiently small (less than some $\delta > 0$).

Formal Definition of a Limit

Let c and L be real numbers. The function **f has a limit L as x approaches c** if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

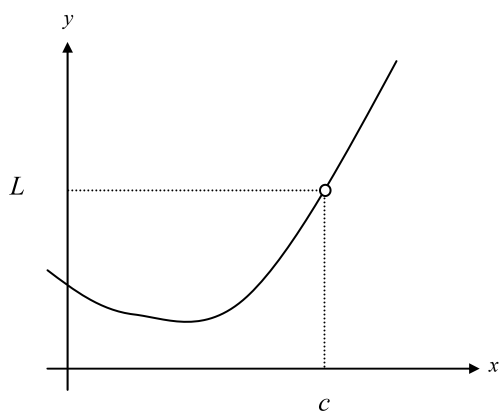
We write

$$\lim_{x \to c} f(x) = L$$

(Just for fun ☺) Symbolically this can be written as follows:

$$\lim_{x \to c} f(x) = L \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0) \ni (0 < |x - c| < \delta) \Rightarrow (|f(x) - L| < \varepsilon)$$

Example: Consider the following function. Graphically show the definition of a limit.



<u>*When Limits Do Not Exist*</u>

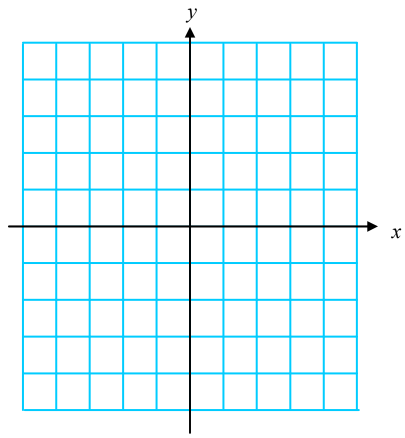
If there does not exist a <u>number</u> L satisfying the condition in the definition, then we say the $\lim_{x \to c} f(x)$ does not exist.

Limits typically fail for three reasons:

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

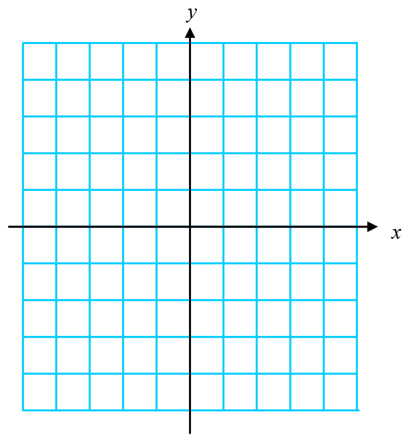
Example: Investigate (use a graph and/or table) the existence of the following limits.

(a) $\lim_{x \to 0} \frac{|x|}{x}$



x	–0.5	–0.25	–0.1	–.01	–.001	0	.001	.01	.1	.25	.5
$f(x)$											

(b) $\lim_{x \to 1} \frac{1}{(x-1)^2}$



x	0	.5	.9	.99	.999	1	1.001	1.01	1.1	1.5	2
$f(x)$											

(c) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ … Try graphing this on your calculator.

First convince yourself that as you move to the right in the chart below x is actually getting closer and closer to 0.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\frac{2}{13\pi}$	As $x \to 0$
$f(x)$								

Properties of Limits

For many "well – behaved" functions, evaluating the limit can be done by direct substitution. That is,

$$\lim_{x \to c} f(x) = f(c)$$

Such *well – behaved* functions are **continuous at *c***. We will study continuity of a function in §2.3.
The following theorems describe limits that can be evaluated by direct substitution.

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

$\lim_{x \to c} b = b$

$\lim_{x \to c} x = c$

$\lim_{x \to c} \left[f(x) \pm g(x) \right] = L \pm K$

$\lim_{x \to c} \left[f(x) \cdot g(x) \right] = LK$

$\lim_{x \to c} \left[b \cdot f(x) \right] = bL$

$\lim_{x \to c} \left[f(x) \right]^{r/s} = L^{r/s}$
provided r and s are integers and $s \neq 0$

$\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$; provided $K \neq 0$

You should realize that all the properties in the box above basically say that you can evaluate a limit as x approaches c by plugging c into the equation.

Example: Find each limit

(a) $\lim_{x \to 1} \left(-x^2 + 1 \right)$

(b) $\lim_{x \to 4} \sqrt[3]{x+4}$

(c) $\lim_{x \to 3} \dfrac{\sqrt{x+1}}{x-4}$

(d) $\lim_{x \to 7} \sec\left(\dfrac{\pi x}{6} \right)$

Example: Use the given information to evaluate the limits: $\lim_{x \to c} f(x) = 2$ and $\lim_{x \to c} g(x) = 3$

(a) $\lim_{x \to c} \left[5g(x) \right]$

(b) $\lim_{x \to c} \left[f(x) + g(x) \right]$

(c) $\lim_{x \to c} \left[f(x) g(x) \right]$

(d) $\lim_{x \to c} \dfrac{f(x)}{g(x)}$

A Strategy for Finding Limits

If a limit cannot be found using direct substitution, then we will use other techniques to evaluate the limit.

♬: Keep in mind that some functions do not have limits.

If direct substitution yields the meaningless result $\frac{0}{0}$, then you ***cannot determine*** the limit in this form.

The expression that yields this result is called an **Indeterminate Form**.
When you encounter this form, you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *cancel like factors*, and a second way is to *rationalize the numerator*.

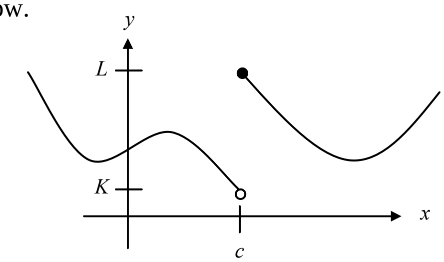
Example: Find the limit: $\lim_{x\to -1} \frac{2x^2 - x - 3}{x+1}$

Example: Find the limit (if it exists): $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$

Example: Find the limit (if it exists): $\lim_{x\to 0} \frac{\left[1/(x+4)\right]-(1/4)}{x}$

One – Sided Limits

One of the reasons a limit did not exist is because the function approached a different value from the left than it did from the right. Suppose we have the graph below.



Earlier, we would have said that the limit as x approaches c does not exist because as x approaches c from the left, the function approaches K, and as x approaches c from the right, the function approaches L. However, sometimes we are interested in what the function approaches as x approaches only from the right or left of c. We can say this using the following notation:

$\lim_{x \to c^+} f(x) = L$ … "the limit of $f(x)$ as x approaches c from the right is L"
$\lim_{x \to c^-} f(x) = K$ … "the limit of $f(x)$ as x approaches c from the left is K"

Thus, we can say that the limit of a function as x approaches any number c exists if and only if the limit as x approaches c from the right is equal to the limit as x approaches c from the left. Using limit notation we have

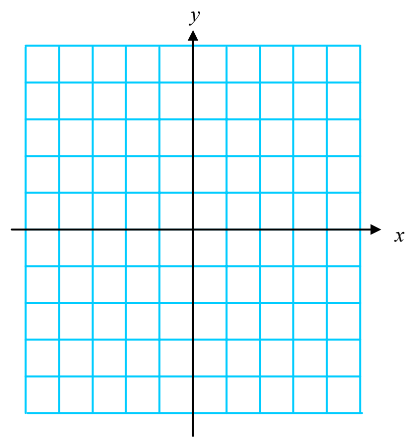
$$\lim_{x \to c} f(x) \text{ exists } \Leftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

Example: Let $f(x) = \text{int } x$ … this is the greatest integer function.

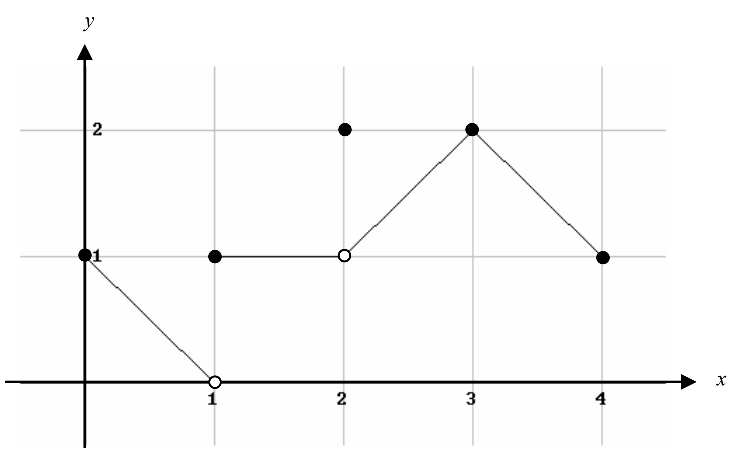
a) Graph $f(x)$

b) Find $\lim_{x \to 2^+} f(x)$.

c) Find $\lim_{x \to 2^-} f(x)$



Example: Find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, and $\lim_{x \to c} f(x)$ for c = 0, 1, 2, 3, 4. If the limit does not exist, explain why.



The Sandwich Theorem (a.k.a. The Squeeze Theorem)

Example: Investigate the $\lim_{x \to 0} \frac{\sin x}{x}$ by sketching a graph and making a table.

You must understand that while using a graph and/or a table, we may be able to determine what a limit is, we have not proved it until we algebraically confirm the limit is what we think it is. The proof of the above limit requires the use of the sandwich theorem.

The Sandwich Theorem

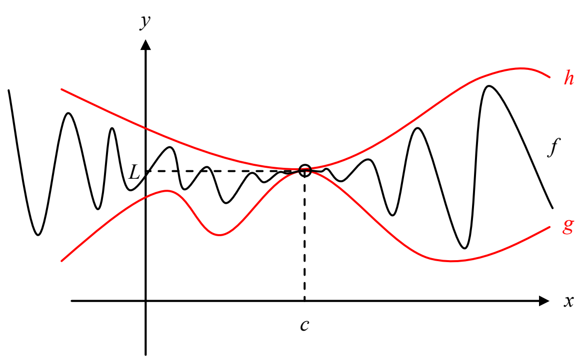
If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about *c*, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

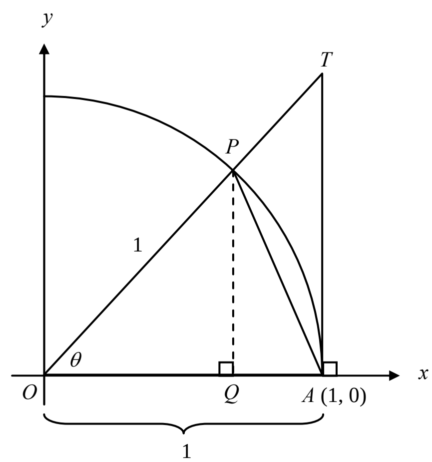
$$\lim_{x \to c} f(x) = L.$$

In other words, if we "sandwich" the function *f* between two other functions *g* and *h* that both have the same limit as *x* approaches *c*, then *f* is "forced" to have the same limit too.



Example: Prove that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. To do this, we are going to use the figure below. Admittedly, the toughest part of using the sandwich theorem is finding two functions to use as "bread" ☺.

First, we need to find $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta}$. In order to do this we need to restrict θ so that $0 < \theta < \frac{\pi}{2}$. Why are we able to do this?



a) Find the area of $\triangle OAP$.

b) Find the area of sector *OAP*.

c) Find the area of $\triangle OAT$.

d) Set up an inequality with the three areas from parts *a*, *b*, and *c*.

e) Divide all three parts by $\frac{1}{2}\sin\theta$. Why do the inequality signs stay the same?

f) Make the middle term $\frac{\sin\theta}{\theta}$. *Hint*: If your middle term doesn't look anything like this, start over! ☺

g) Use the Sandwich Theorem to show that $\lim_{x \to 0^+} \frac{\sin\theta}{\theta} = 1$.

h) Show that $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function.

i) Since $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function, what can you conclude about $\lim_{x \to 0^-} \frac{\sin\theta}{\theta}$?

j) Explain why we can conclude that $\lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$.

Example: Find $\lim_{☺\to 0}\frac{\sin ☺}{☺}$

Example: Find $\lim_{x\to 0}\frac{\sin 5x}{x}$

Example: Find $\lim_{x\to 0}\frac{\sin x}{5x}$

Example: Find $\lim_{x\to 0}\frac{\sin 5x}{4x}$

Example: Find $\lim_{x\to 0}\frac{\sin x}{5x^2+x}$

Example: Find $\lim_{x\to 0}\frac{\tan x}{x}$