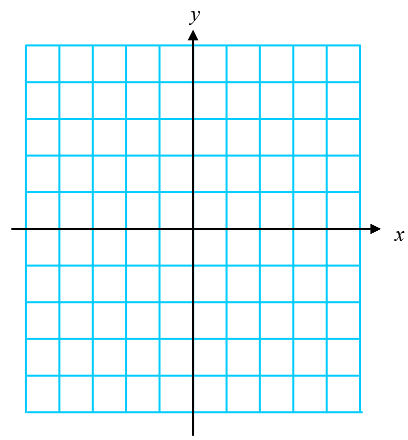
2.2 LIMITS INVOLVING INFINITY

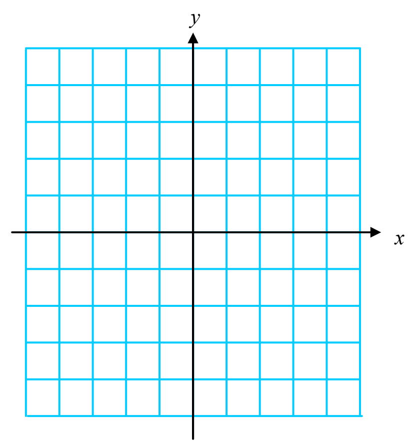
We are going to look at two kinds of limits involving infinity. We are interested in determining what happens to a function as x approaches infinity (in both the positive and negative directions), and we are also interested in studying the behavior of a function that approaches infinity (in both the positive and negative directions) as x approaches a given value.

Finite Limits as $x \to \pm\infty$

Example: Investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{2x-1}{x+3}$.



Example: Investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{1}{x}$.



Both of these graphs have horizontal asymptotes. Are you able to determine what they are from the graphs?

Definition: Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

In the last example, we saw that $\lim_{x\to\infty} \dfrac{1}{x} = 0$. Using this limit and the properties of limits, we can find the limits of other functions as x approaches infinity.

Example: $\lim_{x\to\infty}\left(5 - \dfrac{2}{x^2}\right) =$

Example: Find each of the limits. Determine whether or not there are any horizontal asymptotes. If so, where?

a) $\lim_{x\to\infty} \frac{2x+5}{3x^2-6x+1}$

b) $\lim_{x\to-\infty} \frac{2x+5}{3x^2-6x+1}$

c) $\lim_{x\to\infty} \frac{2x^2-3x+5}{x^2+1}$

d) $\lim_{x\to-\infty} \frac{2x^2-3x+5}{x^2+1}$

e) $\lim_{x\to\infty} \frac{x^4+x^3+9}{3x-3}$

In all of the examples above, if there was a limit as x approached positive infinity, the limit as x approached negative infinity was the same. Thus, there was one horizontal asymptote, and the function approached this same asymptote in both directions. However, certain functions, especially irrational functions, have more than one horizontal asymptote.

Example: Investigate $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ and $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

<u>*End Behavior Models*</u>

Definition: End Behavior Model

The function g is

(a) a **right end behavior model** for f if and only if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$.

(b) a **left end behavior model** for f if and only if $\lim_{x\to-\infty}\frac{f(x)}{g(x)}=1$.

We can use the end behavior models of *rational* functions to identify any horizontal asymptotes.

Example: Go back and look at the last six examples. Find the end behavior models for each.

For rational functions we have the following results. If $f(x)=ax^m+\cdots$ and $g(x)=bx^n+\cdots$ then $\frac{f(x)}{g(x)}$ takes on three different forms.

	End Behavior Model	End Behavior Equation	Asymptote
$m=n$			
$m<n$			
$m>n$			

Example: In the last section we proved that $\lim_{x\to0}\frac{\sin x}{x}=1$. Investigate $\lim_{x\to\infty}\frac{\sin x}{x}$.

It should come as no surprise that the proof of this limit involves the Sandwich Theorem. ☺

The limit properties that we used when x approaches c are still valid as x approaches infinity.

Example: $\lim_{x\to\infty}\left(\frac{2}{x}+1\right)\left(\frac{5x^2-1}{x^2}\right)$

Example: $\lim_{x\to\infty}\frac{x\sin x+2\sin x}{2x^2}$

<u>*Infinite Limits as $x \to a$*</u>

A second type of limit involving infinity is to determine the behavior of the function as x approaches a certain value when the function increases or decreases without bound.

Example: Investigate $\lim_{x\to 0^+}\frac{1}{x}$ and $\lim_{x\to 0^-}\frac{1}{x}$.

In the last example we call the line $x = 0$ a vertical asymptote.

<u>*Definition: Vertical Asymptote*</u>

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x)=\pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x)=\pm\infty$$

Please NOTE: Infinity is NOT a number, and thus the limit FAILS to exist in both of these cases. If this seems confusing, then use the notation as $x \to a$ (from the right or left), then the function $f(x) \to \pm\infty$.

Example: Find the vertical asymptotes of $f(x)$. Describe the behavior of $f(x)$ to the left and right of each asymptote.

a) $f(x)=\frac{x^2-1}{2x+4}$

b) $f(x)=\frac{1-x}{2x^2-5x-3}$

c) $f(x)=\frac{x-2}{3x^2-5x-2}$

Example: Sketch the function that satisfies the stated conditions.

$\lim_{x \to 1} f(x) = 2$

$\lim_{x \to 5^-} f(x) = \infty$

$\lim_{x \to 5^+} f(x) = \infty$

$\lim_{x \to \infty} f(x) = -1$

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to -2^-} f(x) = \infty$

$\lim_{x \to -2^+} f(x) = -\infty$

Example: Sketch the function that satisfies the stated conditions.

$\lim_{x \to 2} f(x) = -1$

$\lim_{x \to 4^+} f(x) = -\infty$

$\lim_{x \to 4^-} f(x) = \infty$

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = 2$