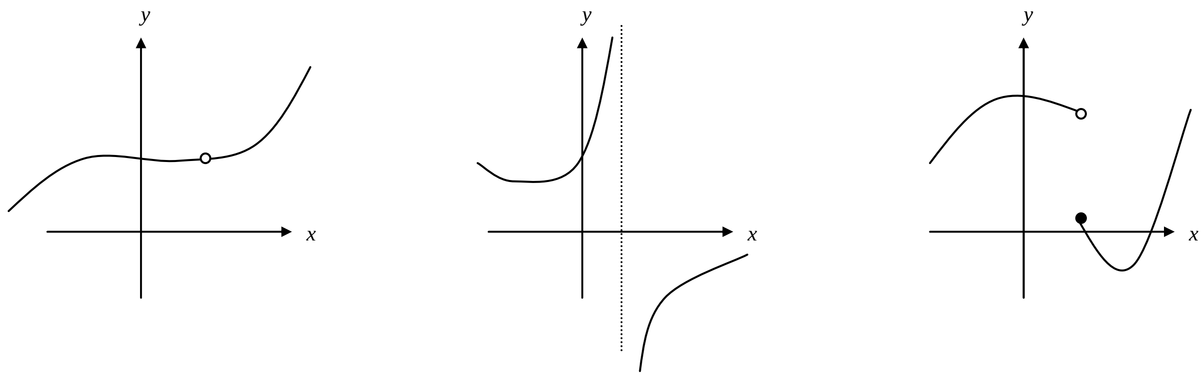
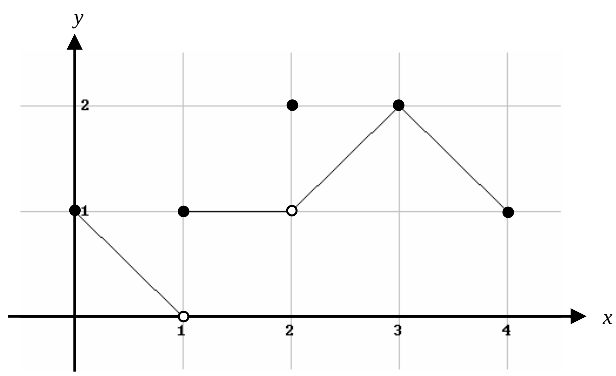
2.3 CONTINUITY

On page 5 of your notes from §2.1 we referred to "well behaved" functions. "Well behaved" functions allowed us to find the limit by direct substitution. In this section we will discuss continuity at a point, continuity on an interval, and the different types of discontinuities.

In non – technical terms, a function is continuous if you can draw the function "without ever lifting your pencil". The following graphs demonstrate three types of discontinuous graphs.



Let's go back to the example we used in §2.1 when we discussed one – sided limits.



Example: Find the points at which the function above is continuous, and the points at which it is discontinuous.

Example: For each point of discontinuity, c, find $f(c)$, $\lim_{x\to c^+} f(x)$, $\lim_{x\to c^-} f(x)$, and $\lim_{x\to c} f(x)$ if they exist.

Definition: *Continuity at a Point*

Interior Point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

♬: This last statement implies that $\lim_{x \to c} f(x)$ exists. This limit only exists if the limit from the left and right of c are equal! It also implies that the function value at c … $f(c)$ exists.

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

Discontinuities: Removable versus Non-Removable

To say a function is discontinuous is not sufficient. We would like to know what type of discontinuity exists. If the function is not continuous, but I could make it continuous by appropriately defining or redefining $f(c)$, then we say that f has a **removable discontinuity**. Otherwise, we say f has a **non-removable discontinuity**.

Once again, informally we say that f has a removable discontinuity if there is a "hole" in the function, but f has a non-removable discontinuity if there is a "jump" or a vertical asymptote.

All polynomials are continuous. For rational functions, we try to algebraically "remove" the discontinuity by canceling factors found in both the denominator and the numerator if possible.

Example: Which (if any) of the three graphs at the top of the other side of the paper have a removable discontinuity?

Example: Discuss the continuity of each function

(a) $f(x) = \dfrac{1}{x-1}$

(b) $g(x) = \dfrac{2x^2 + x - 6}{x+2}$

(c) $h(x) = \begin{cases} -2x+3 & ; x < 1 \\ x^2 & ; x \geq 1 \end{cases}$

Example: Determine the value of c such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+3, & x \leq 2 \\ cx+6, & x > 2 \end{cases}$$

Properties of Continuity

Since continuity is defined using limits, the properties of limits carry over into continuity.

Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Constant multiple: bf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$; $g(c) \neq 0$

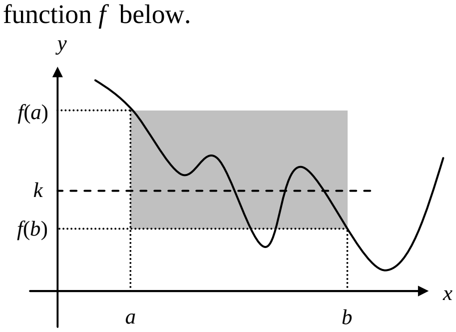
The Intermediate Value Theorem

If f is continuous on the closed interval [a, b] then f takes on every value between $f(a)$ and $f(b)$. Suppose k is any number between $f(a)$ and $f(b)$, then there is at least one number c in [a, b] such that $f(c) = k$.

♫: The Intermediate value theorem tells you that at least one c exists, but it does not give you a method for finding c. This theorem is an example of an *existence theorem*.

Example: In the Intermediate Value Theorem, which axis is k on? What about c?

Example: Consider the function f below.



- f is continuous on [a, b]
- $f(b) < k < f(a)$
- In this example, if $a < c < b$, then there are ________ c's such that $f(c) = k$

Example: Is there any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.

Example: Let $f(x) = \frac{x^2 + x}{x-1}$. Verify that the Intermediate Value Theorem applies to the interval $\left[\frac{5}{2}, 4\right]$ and find the value of c guaranteed by the theorem if $f(c) = 6$.

Problems to Think About

1. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following:

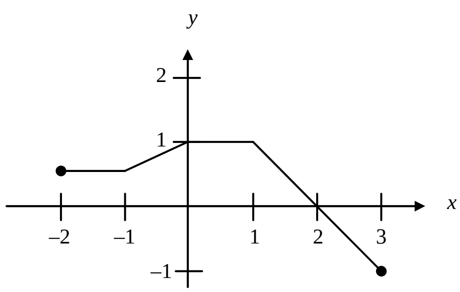
   a) a function with a nonremovable discontinuity at $x = 2$
   b) a function with a removable discontinuity at $x = -2$.
   c) a function that has both of the characteristics described in parts *a* and *b*.

2. Give a written explanation of why the function $f(x) = x^2 - 4x + 3$ has a zero in the interval [2, 4]

3. At 8:00 AM on Saturday a man begins running up the side of a mountain to his weekend campsite. On Sunday morning at 8:00 AM he runs back down the mountain along the same trail. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at EXACTLY the same time on Saturday. Prove he is correct. [*Hint*: Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]

4. True or False? If false, explain/show why. If $p(x)$ is a polynomial, then the function given by $f(x) = \frac{p(x)}{x-1}$ has a vertical asymptote at $x = 1$.

5. Use the graph of the function f below to sketch the graph of $g(x) = {}^{1}\!/_{f(x)}$ on the interval [–2, 3].



6. True or False? If false, explain/show why. If $f(c) = L$, then $\lim_{x \to c} f(x) = L$.

7. True or False? If false, explain/show why. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

8. True or False? If false, explain/show why. If $\lim_{x \to c} f(x) = L$ and $f(c) = L$, the f is continuous at c.