3.7 IMPLICIT DIFFERENTIATION

Implicit and Explicit Functions

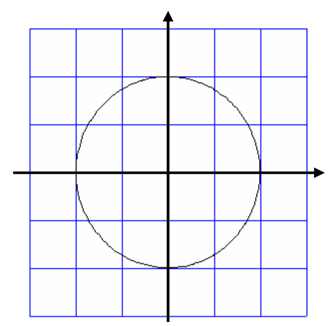
Suppose your boss says, "I have had it with your incompetence. You've screwed up everything we've ever given you to do! The entire company is on the brink of bankruptcy, single-handedly thanks to you! Now, clean out your desk and don't show your face here again!"
You might argue that the boss never actually said, "You're fired." There was not explicit dismissal ever stated, but you have to face the facts: It was certainly implied!

Equations can be written with implied meanings as well. For instance, the equation $x + 2y - 3 = 0$, implies that *y* is a function of *x*, even though it is not written in the form $y = -\frac{1}{2}x + \frac{3}{2}$. Up to this point in this class we have been using functions of *x* expressed in the form $y = f(x)$ such as

$$y = \frac{x+1}{x+2} \qquad \text{or} \qquad y = \sin x.$$

An equation of this form is said to define *y* as an *explicit* function of *x*.
If we have an equation that involves both *x* and *y* in which *y* has not been solved for *x*, then we say the equation defines *y* as an *implicit* function of *x*. In this case, we may (or may not) be able to solve for *y* in terms of *x* to obtain an explicit function (or possibly several functions).

Example: Solve the equation $x^2 + y^2 = 4$ to obtain *y* written as an explicit function of *x*. The graph of this equation is a circle. What is the graph of each explicit function?



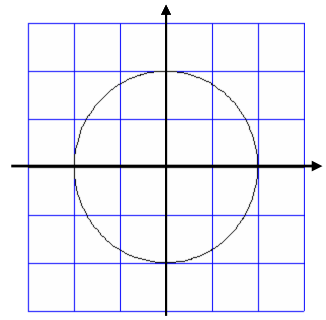
If we have *y* written as an explicit function of *x*, $y = f(x)$, then we know how to compute the derivative $\frac{dy}{dx}$. For an equation which defines *y* as an implicit function of *x*, we can compute the derivative $\frac{dy}{dx}$ without solving for *y* in terms of *x* with the following procedure. **The key to this entire procedure is to remember that even though you did not (or cannot) write *y* as a function of *x*, *y* is implicitly defined as a function of *x*.**

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation **with respect to *x*.** Remember, *y* is a function of *x* (use the Chain Rule)
2. Collect all $\frac{dy}{dx}$ terms on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation.
4. Solve for $\frac{dy}{dx}$. (It is okay to have both *x*'s and *y*'s in your answer)

To find $\frac{dy}{dx}$ at a given point, plug both the *x* and *y* value into the equation you obtained in step 4.

Example: Find $\frac{dy}{dx}$ for the circle $x^2 + y^2 = 4$:

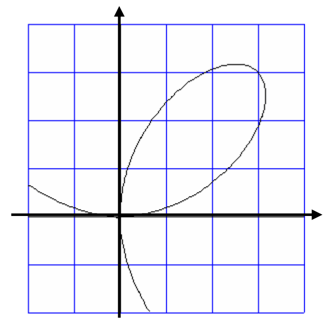


a) by differentiating the explicit functions of x

b) by differentiating implicitly

c) Find where the derivative is positive, where it is negative, where it is zero, and where it is undefined.

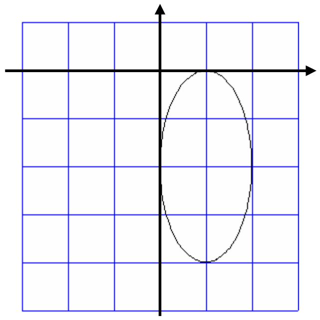
Example: Given the curve $x^3 + y^3 = 6xy$ (shown to the right).



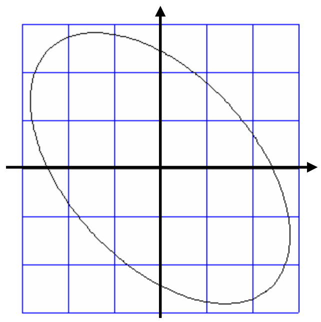
a) Find $\frac{dy}{dx}$.

b) Find the equation of the tangent line and normal (perpendicular) line to the graph at the point $\left(\frac{4}{3}, \frac{8}{3}\right)$.

Example: Find the points at which the graph of $4x^2 + y^2 - 8x + 4y + 4 = 0$ has a vertical tangent line.

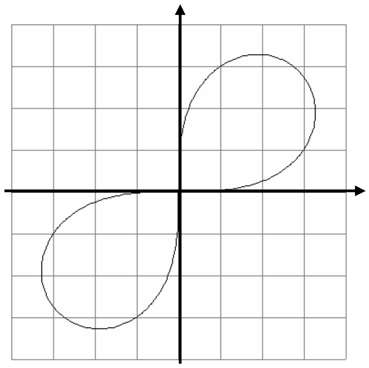


Example: Find the point(s) (if any) of horizontal tangent lines: $x^2 + xy + y^2 = 6$



Example: Find $\frac{dy}{dx}$ at (0, 0) of the function $\tan(x + y) = x$.

Example: Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point (3, 1).



Example: [2004 AP Calculus AB Free Response Question #4 … No Calculator Allowed]
Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

a) Show $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$

b) Show that there is a point *P* with *x* – coordinate 3 at which the line tangent to the curve *P* is horizontal. Find the *y* – coordinate of *P*.

c) Find the value of $\frac{d^2y}{dx^2}$ at the point *P* found in part *b*. ~~Does the curve have a local maximum, a local minimum, or neither at the point *P* ? Justify your answer.~~