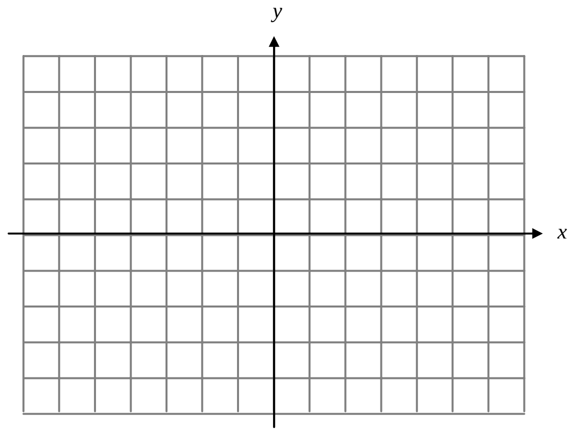
4.2 MEAN VALUE THEOREM

The Mean Value Theorem is considered by some to be the most important theorem in all of calculus. It is used to prove many of the theorems in calculus that we use in this course as well as further studies into calculus.

Example: Graph the points (–6, 4) and (5, –4) on the grid below.



Example: Draw a non – linear function, passing through the points above, that is continuous on the closed interval [–6, 5] and differentiable on the open interval (–6, 5).

Example: Draw a line between the points (–6, 4) and (5, –4). Calculate the slope of this line.

Example: Are there any other points on your function, where the tangent line has the same slope as the line joining (–6, 4) and (5, –4)? Sketch these tangent lines.

The Mean Value Theorem

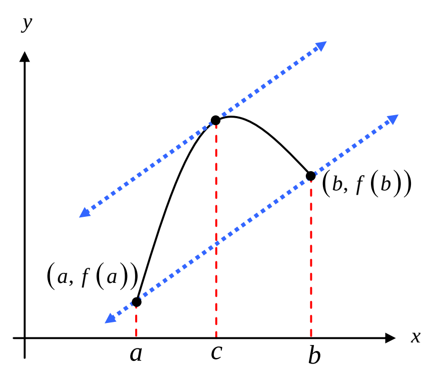
If f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then ***<u>there exists</u>*** a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Just like the Intermediate Value Theorem, this is an *existence theorem*. The Mean Value Theorem does not tell you what the value of c is, nor does it tell you have many exist. Again, just like the Intermediate Value Theorem, you must keep in mind that c is an x – value.

Also, the hypothesis of the Mean Value Theorem (MVT) is highly important. If any part of the hypothesis does not hold, the theorem cannot be applied.

Basically, the Mean Value Theorem says, that the average rate of change over the entire interval is equal to the instantaneous rate of change at some point in the interval.



Example: A plane begins its takeoff at 2:00 pm on a 2500-mile flight. The plane arrives at its destination at 7:30 pm (ignore time zone changes). Explain why there were at least two times during the flight when the speed of the plane was 400 miles per hour.

Example: Apply the Mean Value Theorem to the function on the indicated interval. In each case, make sure the hypothesis is true, then find all values of *c* in the interval that are guaranteed by the MVT.

a) $f(x) = x(x^2 - x - 2)$ on the interval [–1, 1]

b) $f(x) = \frac{x+1}{x}$ on the interval [0.5, 2].

c) $f(x) = -2x^2 + 14x - 12$ on the interval [1, 6]

The last example is a special version of the Mean Value Theorem called Rolle’s Theorem. In fact, the proof of the Mean Value Theorem can be done quite easily, if you prove Rolle’s Theorem first. Rolle’s Theorem basically states that if the function is continuous on the closed interval and differentiable on the open interval AND the values of the function at the endpoints are equal, then there must exist at least one point in the interval where the derivative is zero.

Consequences of the Mean Value Theorem

While the Mean Value Theorem is used to prove a wide variety of theorems, we will be focusing on the results and/or consequences of the Mean Value Theorem. In this section, we will discuss when a function increases and decreases as well as a brief introduction to antiderivatives.

Increasing versus Decreasing

Why mathematicians feel the need to define everything is a mystery you will probably never figure out unless you become one. Then, for some inexplicable reason, you will find yourself questioning the truthfulness of every argument ever made, reducing every argument to its basic foundational vocabulary, and finally analyzing the very soul and fiber of the definitions. With this in mind, we will now define what it means for a function to be increasing and decreasing. (Obviously, we cannot just say that a function is increasing when all the function values get bigger.)

Definitions of Increasing and Decreasing Functions

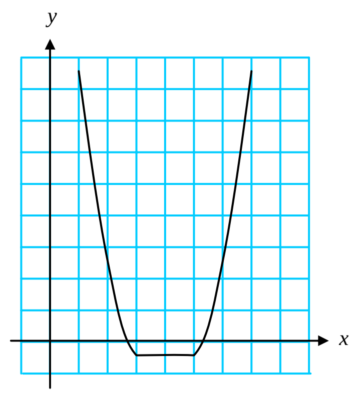
A function *f* is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function *f* is **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

Example: What interval is the function decreasing? increasing? constant?



Example: What is the value of the derivative when the function is decreasing? increasing? constant?

Test for Increasing and Decreasing Functions

Let *f* be a function that is continuous on the closed interval [*a*, *b*] and differentiable on the open interval (*a*, *b*).

1. If $f'(x) > 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is increasing on [*a*, *b*].
2. If $f'(x) < 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is decreasing on [*a*, *b*].
3. If $f'(x) = 0$ <u>for all</u> *x* in (*a*, *b*), then *f* is constant on [*a*, *b*].

The proof of the above concepts comes directly from the Mean Value Theorem.

Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

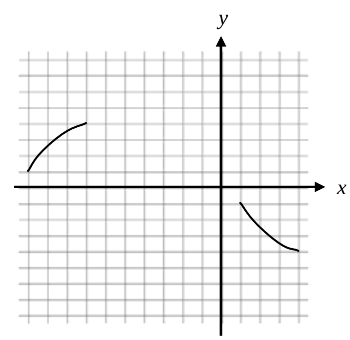
Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing use the following steps:

1. Find the critical points of f in the interval (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at ONE test value in each interval.
3. Use the signs of the derivative to determine whether or not the function is increasing or decreasing.

Example: Find the intervals on which $f(x) = x^3 - 12x - 5$ is increasing or decreasing.

Example: Find the intervals on which $f(x) = \left(x^2 - 9\right)^{2/3}$ is increasing or decreasing.

Example: The figure to the right gives two parts of the graph of a continuous differentiable function f on [–10, 4]. The derivative f' is also continuous.

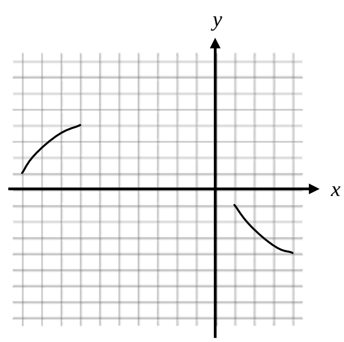


a) Explain why f must have at least one zero in [–10, 4].

b) Explain why f' must also have at least one zero in the interval [–10, 4]. What are these zeros called?

c) Make a possible sketch of the function with one zero of f' on the interval [–10, 4].

d) Make a possible sketch (on the graph below) of the function with two zeros of f' on the interval [–10, 4].

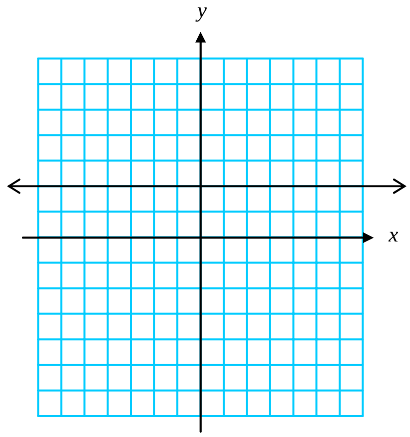


Antiderivatives

Example: Suppose you were told that $f'(x)=2x-1$. What could $f(x)$ possibly be? Is there more than one answer?

Finding the function from the derivative is a process called ***antidifferentiation***, or finding the antiderivative.

Example: Suppose the graph of $f'(x)$ is given below. Draw at least three possible functions for $f(x)$.



The three functions you drew should only differ by a constant. If you let C represent this constant, then you can represent the *family* of all antiderivatives of $f'(x)$ to be $f(x)=2x+C$.

Example: If you were told that $f(3)=-2$, what would the value of C be?

IMPORTANT ♬: If a function has one antiderivative it has many antiderivatives that all differ by a constant. Unless you know something about the original function, you cannot determine the exact value of that constant, but it must be in your answer!

Example: If you know that the acceleration of gravity is $-32\ {}^{ft}\!/\!_{s^2}$, for an falling object, we could write the acceleration of the object at time t as $a(t)=-32$. Find a function for the velocity of the object at time t. What does the constant equal (in words)?

Example: Find a function for the position of the object at time t. What does the constant equal (in words)?

Example: [2004 AP Calculus AB Free Response Question #3 … Calculator Allowed] A particle moves along the *y*-axis so that its velocity at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}(e^t)$. At time $t = 0$, the particle is at $y = -1$.
(Note: $\tan^{-1} x = \arctan x$)

a) Find the acceleration of the particle at time $t = 2$.

b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

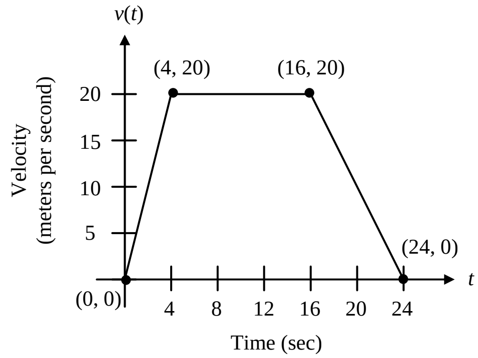
d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

[You can't actually do part d yet … but you should be able to do the following:]

i) Explain what you would need to do in order to find the position at time t = 2 and what information in the original problem is useful (and necessary) for this part of the question.

ii) If you were able to determine position of the particle at t = 2, then you should be able to explain how to determine the direction of the particle at t = 2. For instance, if you knew the position of the particle at t = 2 was –1.361, then how would you determine whether or not the particle was moving toward the origin or away from the origin?

Example: [2005 AB Calculus AP Free Response #5 … No Calculator Allowed] A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity *v*(*t*), in meters per second, is modeled by the piecewise-linear function defined by the graph below.



a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

[Obviously we haven't used this symbol yet, nor have we talked about how to get it … so here's a couple of hints …]

i) If I told you the notation $\int_0^{24} v(t)\,dt$ *only asked you to find the antiderivative of the velocity function, you should be able to* *<u>use correct units</u>.*

ii) If I told you that all the notation $\int_0^{24} v(t)\,dt$ *means for this problem is to find the area under the given curve, you should then be able to answer the question AND* *<u>explain the meaning</u>* *of* $\int_0^{24} v(t)\,dt$.

b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

c) Let *a*(*t*) be the car's acceleration at time *t*, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for *a*(*t*).

d) Find the average rate of change of *v* over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of *c*, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?