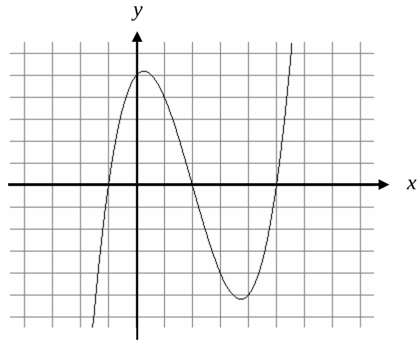
4.3 CONNECTING f' AND f'' WITH THE GRAPH OF f

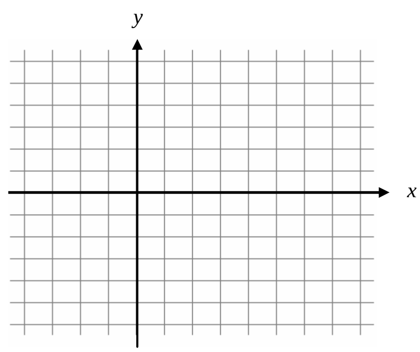
First Derivative Test for Extrema

We have already determined that relative extrema occur at critical points. The behavior of the first derivative before and after those critical points will help determine whether or not the function has a relative maximum or minimum (or neither) at these critical points.

Example: Given the graph of f below, label the relative extrema.



Example: Sketch a graph f', on the axis below. Label the x – values of f' where the extrema occur on f.



The First Derivative Test

Let f be a continuous function, and let c be a critical point.

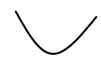
1. If f' changes sign from positive to negative at c, then f has a local maximum value at c.
2. If f' changes sign from negative to positive at c, then f has a local minimum value at c.
3. If f' DOES NOT change signs, then there is no local extreme value at c.

Important ♬: If you are asked to find the absolute maximum (or just a maximum) of a function on a closed interval, you MUST test the endpoints also.

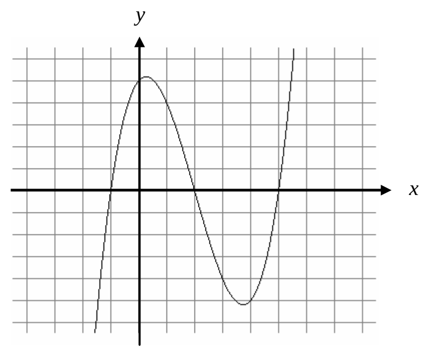
Example: Find where the function $h(x) = x\sqrt{4-x^2}$ is increasing and decreasing, then use the first derivative test to determine any local extrema.

Example: Find where the function $g(x) = x^2 e^x$ is increasing and decreasing, then find any local extrema and absolute extrema.

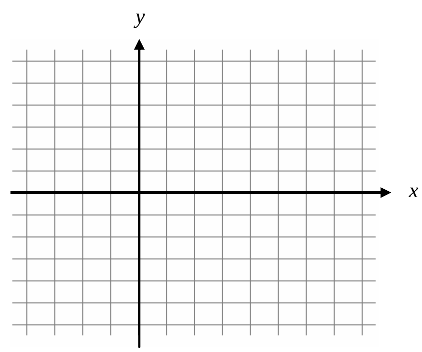
Concavity

Concavity deals with how a graph is curved. A graph that is concave up looks like  , while a graph that is concave down looks like  . We can use the SECOND derivative to determine the concavity of a function.

Example: Using the same graph as our previous example, indicate which portions of the graph are concave up, and which portions are concave down. Label the point where the graph changes concavity, and then sketch the graph of the first derivative <u>on the same graph.</u>



Example: Sketch the graph of the *second* derivative on the graph below. Label the *x*-value where the graph changes concavity.



Example: Using the graphs you sketched, circle the correct word that completes the following statements:

a) When the graph of *f* is increasing, *f* ' is (positive , negative).

b) When the graph of *f* is decreasing, *f* ' is (positive , negative).

c) The graph of *f* is concave upward when *f* " is (positive , negative).

d) The graph of *f* is concave downward when *f* " is (positive , negative).

e) The graph of *f* is concave upward when *f* ' is (increasing , decreasing).

f) The graph of *f* is concave downward when *f* ' is (increasing , decreasing).

Using the results from the last few statements leads us to the following definition.

Definition of Concavity

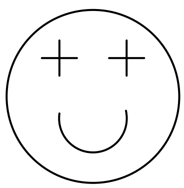
Let $y = f(x)$ be a differentiable function on an interval I. The graph of $f(x)$ is **concave up** on I if f' is increasing on I, and **concave down** on I if f' is decreasing on I.

If the first derivative is increasing, then the second derivative must be __________________. If the first derivative is decreasing, then the second derivative must be __________________. Thus instead of using the definition of concavity to determine whether the function is concave up or down, we can use the following test.

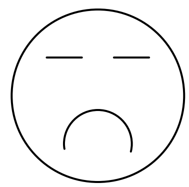
Concavity TEST

The graph of a twice-differentiable function $y = f(x)$ is **concave up** on any interval where $y'' > 0$, and **concave down** on any interval where $y'' < 0$.

The Concavity Test can be summed up by the following pictures … While this is a humorous (and hopefully helpful) way to remember concavity, please understand that this is NEVER to be used as a justification on ANY test!



f'' positive $\Rightarrow$ Concave UP



f'' negative $\Rightarrow$ Concave DOWN

Example: Find the intervals where the function $g(x) = -2x^3 + 6x^2 - 3$ is concave up and concave down.

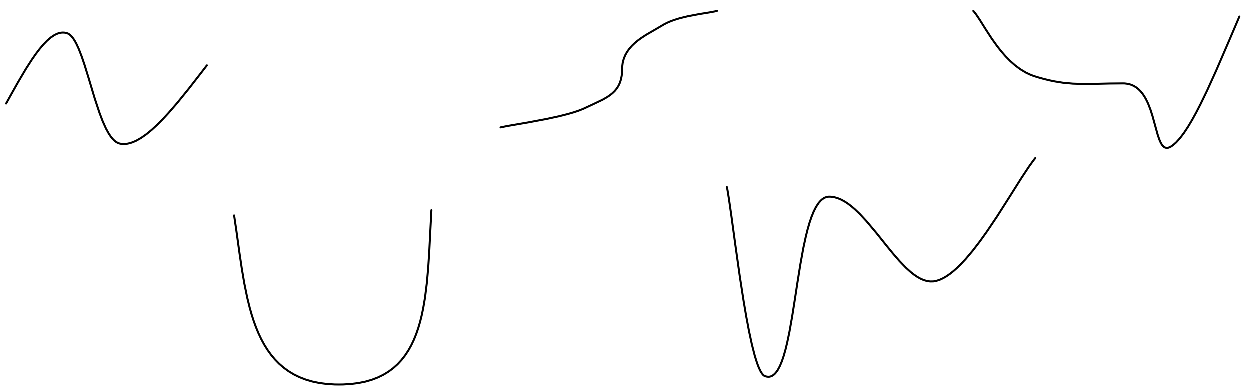
Example: Determine the concavity of $h(x) = x\sqrt{4 - x^2}$.

Points of Inflection

Definition

A point where the graph of a function <u>has a tangent line</u> (even if it's a vertical tangent line) AND where <u>the concavity changes</u> is a **point of inflection**.

Example: Using each picture, estimate each point of inflection, if any, and sketch the tangent line at that point.



Since points of inflection occur when the graph changes ________________, and a graph changes concavity when the ____________________________________ changes from positive to negative (or vice-versa), then if we wanted to find the points of inflection of a graph, we only need to focus on when the second derivative equals 0 (or does not exist)

<u>IMPORTANT ♬</u>: Just because the second derivative equals zero (or does not exist) you are NOT guaranteed that the function has a point of inflection. The second derivative MUST change signs (meaning concavity changed) in order for a point of inflection to exist!

Example: Go back to page 69 of your notes and look at the graph you drew of the second derivative. Estimate the *x*-value of the point of inflection on the original function. Where SHOULD this value be graphed when graphing the second derivative? Fix the graph if necessary.

Example: Find the points of inflection of $g(x) = -2x^3 + 6x^2 - 3$.

Example: Find the points of inflection of $h(x) = x\sqrt{4 - x^2}$.

Second Derivative Test for Extrema

Example: Go back to the original function on page 68. First look at the point where the function had a maximum. Was the graph concave up or down at that point? Secondly, look at the point where the function had a minimum. Was the graph concave up or down at that point?

As long as the function is twice-differentiable (meaning the first derivative is a smooth curve), then we can actually determine whether or not a <u>critical point</u> is a relative maximum or minimum WITHOUT testing values to the right and left of the point. We can use the Second Derivative Test.

Second Derivative Test for Local Extrema

If $f'(c)=0$ (which makes $x = c$ a critical point) AND $\boldsymbol{f''(c)<0}$, then f has a local MAXIMUM at $x = c$.
If $f'(c)=0$ (which makes $x = c$ a critical point) AND $\boldsymbol{f''(c)<0}$, then f has a local MINIMUM at $x = c$.

Important ♬: If the second derivative is equal to zero (or undefined) then the Second Derivative Test is INCONCLUSIVE.

Remember the happy (and sad) faces? If a critical point happens to occur in an interval where the graph of the function is CONCAVE UP, then that critical point is a relative MINIMUM. If a critical point happens to occur in an interval where the graph of the function is CONCAVE DOWN, then that critical point is a relative MAXIMUM.

Example: Use the Second Derivative Test to identify any relative extrema for the function $g(x)=-x^4+4x^3-4x+1$.

<u>NOTES FOR OPTIMIZATION PROBLEMS:</u>

Whenever you are required to Maximize or Minimize a function, you MUST <u>justify</u> whether or not your answer is actually a maximum or a minimum. You may use the FIRST DERIVATIVE TEST (testing points to the left and right of the critical points in the first derivative to see if the sign of the first derivative changes from positive to negative or vice-versa), or the SECOND DERIVATIVE TEST (plugging in the critical points to the second derivative to see if the critical points occur when the original function was concave up or down).

ALWAYS REMEMBER that both of these tests are checking for relative extrema. If you have a CLOSED interval, you must check the endpoints to make sure the absolute maximum or minimum values do not happen to occur there.

Example: [2005 AP Calculus AB Free Response #4 … No Calculator Allowed]

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$	3	$3<x<4$
$f(x)$	–1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	–3	Negative
$f''(x)$	–2	Negative	0	Positive	DNE	Negative	0	Positive

Let f be a function that is continuous on the interval [0, 4]. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exist at $x = 2$.

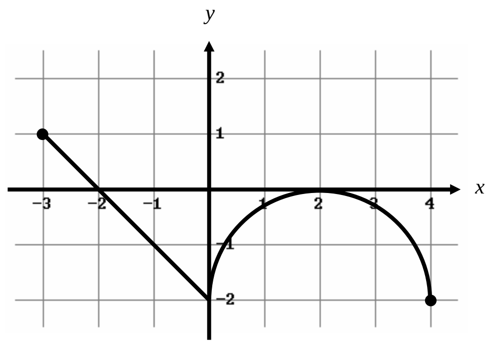
a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

b) On the axes provided, sketch the graph of a function that has all the characteristics of f.

~~c) Let g be the function defined by $g(x)=\int_1^x f(t)\,dt$ on the open interval (0, 4). For $0 < x < 4$, find all values of x at which g has a relative extremum. Determine whether g has a relative maximum or a relative minimum at each of these values. Justify your answer.~~

~~d) For the function g defined in part c, find all values of x, for $0 < x < 4$, at which the graph of g has a point of inflection. Justify your answer.~~

Example: [2003 AP Calculus AB Free Response #4 … No Calculator Allowed] Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown below.



Graph of f'

a) On what intervals, if any, is f increasing? Justify your answer.

b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

c) Find an equation for the line tangent to the graph of f at the point (0, 3).

~~d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.~~