4.5 LINEARIZATION AND NEWTON'S METHOD

Linearization

The goal of linearization is to approximate a curve with a line. Why? Because it's easier to use a line than a curve! The basic idea of linearization is to find the equation of the tangent line at a certain point, and use the tangent line to estimate the desired value of the original function.

Example: Consider $f(x)=\sqrt{x}$. We all know that $f(4) = 2$, but without a calculator, what is $f(4.1)$?

a) Find the equation of the tangent line for $f(x)$ when $x = 4$, call it $L(x)$.

b) We say that $L(x)$ is approximately the same as $f(x)$ centered at $x = 4$. Use $L(x)$ to approximate $f(4.1)$.

c) Use a calculator to approximate $f(4.1)$. Are you close?

Example: [1969 Multiple Choice AB #36] The approximate value of $y=\sqrt{4+\sin x}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$, is

A) 2.00 B) 2.03 C) 2.06 D) 2.12 E) 2.24

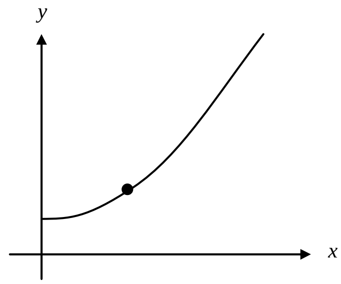
Example: [1973 Multiple Choice AB #44] For small values of h, the function $\sqrt[4]{16+h}$ is best approximated by which of the following?

A) $4+\frac{h}{32}$ B) $2+\frac{h}{32}$ C) $\frac{h}{32}$ D) $4-\frac{h}{32}$ E) $2-\frac{h}{32}$

Differentials

Approximations aren't exact! (Aren't you glad you woke up this morning to hear that enlightening bit of information?!) If we use a line to approximate a curve, it gives us a good estimate, as long as we don't go too far away from the center point. Wouldn't it be nice if we knew how far off our approximation is going to be? Well, whether you are excited about this or not, here we go!

Example: Consider the function f below. Label the point $(c, f(c))$, and draw the tangent line at that point.



Example: What is the equation of the *tangent line* you drew?

Move a "small" distance to the right of c. Normally, we would call this distance Δx, but when Δx is very small, we will instead use the notation dx, the **differential of *x***.

Example: What is the **function** value at this point?

Example: What is the value of this point on the *tangent line* (when $x = c + dx$) ?

Example: How much did the y-values <u>ACTUALLY</u> change?
[Find Δy on the **function**]

Example: How much did the y-values <u>APPROXIMATELY</u> change?
[Find dy (the differential of y … a small change in y} on the *tangent line*]

In other words, if we were to use any value of x, the approximate change in y after a small change in x would be written

$$dy = f'(x)dx$$

This should look VERY familiar … What differentials allow us to do is to say that if the ratio of the differentials exists, it will be equal to the derivative. It allows us to write $\frac{dy}{dx}$ as the derivative of y with respect to x, but use the dy and dx as separate terms. Finding a differential is very similar to finding a derivative.

Example: Find the differential dy if $y = x^3 - 5x$

Example: Find $d(\cos 5x)$.

Since *dy* is the approximate change in the *y* values when *x* is changed a small amount, we can use differentials to estimate the change in other problems if we know the small change in *x*.

Example: Find the differential *dy* when *dx* = 0.01 and *x* = 2, if $y = x^5 - 4x^3$. Explain what you've found.

Example: Find the differential *dy* when *dx* = –0.2 and *x* = 1, if $y = x^2 e^x$. Explain what you've found.

Example: Without a calculator, use differentials to approximate $\sqrt{4.2}$.

As we move from a point *c* to a nearby point *c* + *dx*, we can describe the change in *f* three ways:

	TRUE	**ESTIMATED**
Absolute change		
Relative change		
Percentage change		

Example: The range *R* of a projectile is $R = \frac{{v_0}^2}{32}(\sin 2\theta)$, where ${v_0}^2$ is the initial velocity in feet per second and θ is the angle of elevation. If $v_0 = 2200$ feet per second and θ changed from 10° to 11°, use differentials to approximate the change in the range.

Example: The measurement of a side of a square is found to be 15 centimeters. The possible error in measuring the side is 0.05 centimeter.

a) Approximate the percent error in computing the area of the square.

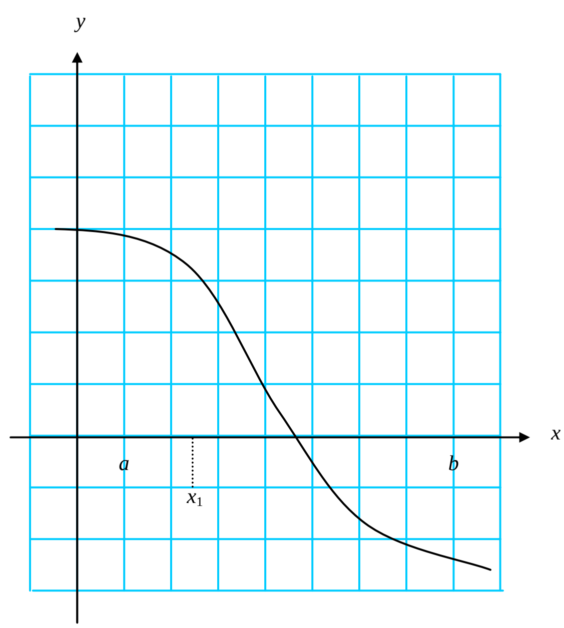
b) Estimate the maximum allowable percent error in measuring the side if the error in computing the area cannot exceed 2.5%.

Example: A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5°. How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

Newton's Method (no longer part of the AB curriculum) … time permitting …

Newton's method is a process using linearization to approximate (amazingly accurately) the zeroes of a function.

Example: Consider a continuous, differentiable function such that $f(a) > 0$ and $f(b) < 0$. What does the Intermediate Value theorem guarantee must occur between *a* and *b*?



Newton's method is based on the assumption that the tangent line at a point crosses the *x* axis at *about* the same place as the function. Since it is relatively easy to calculate the *x* – intercept of a line, we use this line to create a new estimate of the zero.

Example: Locate the function value at the point x_1. Draw a tangent line at that point.

Example: **Label** the *x* – intercept of the tangent line x_2. **Locate** the function value at x_2. **Draw** the tangent line at this point and **find** the *x* –intercept of the line.

Example: Is this point close to the point where the original function crosses the *x* – axis?

Example: Repeat the process one more time. How close do you get to the actual *x* – intercept?

Example: Can you think of a situation where this method would NOT work?