4.6 RELATED RATES

We have been taking derivatives with respect to x for almost the entire year. When even took derivatives of y implicitly, but as a function of x. Suppose a particle is moving along the curve $y = x^2$. As it moves, the x and y coordinates are changing at the same time. If we viewed their change as a function of time, then we could represent the rate of change in the y values with respect to time as $\frac{dy}{dt}$ and the rate of change in the x values with respect to time as $\frac{dx}{dt}$. We find these derivatives in the same way you found derivatives implicitly.

Example: Suppose both y and x are differentiable functions of t. Differentiate $y = x^2$ with respect to t.

Example: If you wanted to solve for $\frac{dy}{dt}$, what other information would have to be given to you?

Example: Suppose you are told that the particle moving along the curve $y = x^2$ is moving horizontally at 2 cm/s. Find the rate of change in the particle's vertical position at the exact moment the particle is at (3, 9).

In a related rates problem, you have an equation relating two or more things that *change over time*, and we want to find the rate of change (a derivative) of one of these things. It is important to understand that without some conditions given to use, we cannot solve the problem.

Guidelines for Solving Related-Rate Problems

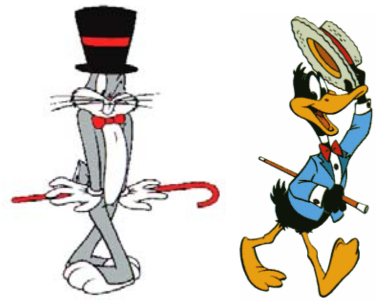
Step 1: Read the problem, really! You'd be amazed how many people skip this step. Then read it again! ☺
Step 2: Draw a diagram showing what's going on. Identify all relevant information and assign variables to what's changing. Use the general case (numbers for values that NEVER change in this situation, and variables for anything that is changing).

♬: Related Rates usually involve motion … any diagram you draw is like a still picture of what is occurring. Any part of your picture that NEVER changes can be labeled with a constant (or number), but any part of your picture that is in motion or is changing MUST be labeled with a variable!

In other words, if the radius of a circle is increasing and you are asked to find the rate of change in the area at the exact moment when the radius is 5 cm, then your diagram would be a circle, but you would NOT label the radius 5 because it is changing … you would label the radius r.

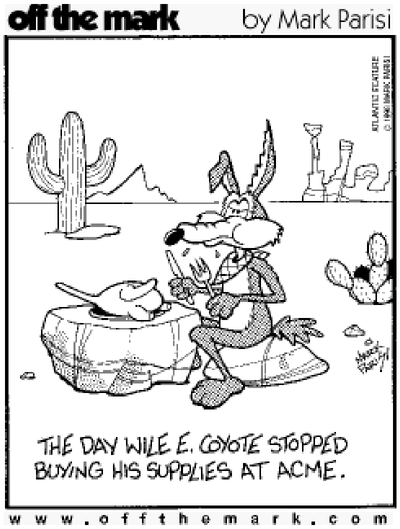
Step 3: Find the equation that gives the relationship between the variables you just named in step 2. This is sometimes the hardest part, but most problems fall into three categories … a triangle that you can use a trigonometric ratio (involving sides and angles), the Pythagorean theorem (involving all 3 sides of a right triangle), or a known formula like Area, Volume, Distance, etc.
Step 4: Find the particular information (values of variables at the exact moment you drew your diagram) for the problem and write it down, and list what you are looking for (normally this would be a derivative).
Step 5: Implicitly differentiate the equation with respect to time. Usually this equation will have at least two derivatives. If it has more than two, be sure you have enough information, or you may have find a relationship between two of the variables, and rewrite the equation in step 3 using this relationship.
Step 6: Plug in the particular information, and solve for the desired quantity. DO NOT DO THIS UNTIL AFTER YOU HAVE TAKEN THE DERIVATIVE!
Step 7: Write down your answer and circle it with your favorite color. (be sure to use correct units)

Example: Bugs and Daffy finished their final act on the *Bugs and Daffy Show* by dancing off the stage with a spotlight covering their every move. If they are moving off the stage along a straight path at a speed of 4 ft/s, and the spotlight is 20 ft away from this path, what rate is the spotlight rotating when they are 15 feet from the point on the path closest to the spotlight?

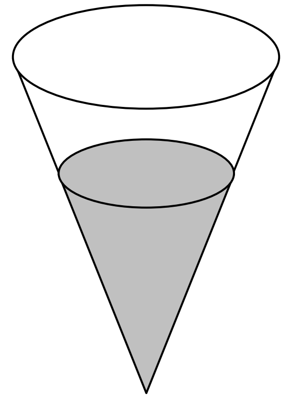


Example: Tweety is resting in a bird house 24 feet off the ground. Using a 26 foot ladder which he leaned against the pole holding the bird house, Sylvester tries to steal the small yellow bird. Tweety's bodyguard, Hector the dog, starts pulling the base of the ladder away from the pole at a rate of 2 ft/s. How fast is the ladder falling when it is 10 feet off the ground?

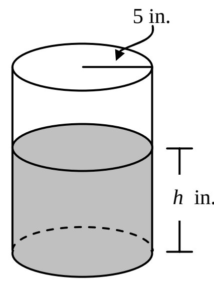
OK … I couldn't find a decent looney tunes picture for the next problem, so I thought I'd just throw in this cartoon (which by the way has nothing to do with related rates!) since I found it looking for any other good pictures. Besides, poor Wile E. Coyote has been working so much this year, it's about time he finally got a good meal. ☺



Example: A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep. The volume of a circular cone with radius *r* and height *h* is given by $V = \frac{1}{3}\pi r^2 h$.



Example: A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure to the right. Let *h* be the depth of the coffee in the pot, measured in inches, where *h* is a function of time *t*, measured in seconds. The volume *V* of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume *V* of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$.) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.



Example: A baseball diamond has the shape of a square with sides 90 feet long. Tweety is just flying around the bases, running from 2^{nd} base (top of the diamond) to third base (left side of diamond) at a speed of 28 feet per second. When Tweety is 30 feet from third base, at what rate is Tweety's distance from home plate (bottom of diamond) changing?