

**5.1 ESTIMATING WITH FINITE SUMS**

*Example:* Suppose from the 2<sup>nd</sup> to 4<sup>th</sup> hour of your road trip, you travel with the cruise control set to exactly 70 miles per hour for that two hour stretch. How far have you traveled during this time?

*Example:* Sketch a graph modeling the situation in the above example. Geometrically, how can we indicate the total distance traveled?

*Example:* What if the velocity was NOT constant. Say, for instance the velocity in miles per hour is given by the function  $v(t) = 10t - t^2$ , where  $t$  is in hours, and we wanted to know the total distance traveled during the first 10 hours. Sketch this graph below. Geometrically speaking, do you think we can find the total distance traveled in the same way as before? Why or why not?

Word of caution to those brave few who are actually reading this ... The following paragraphs are extremely important to the conceptual understanding of what we are about to do in Calculus. However, since you really haven't done anything yet, it might make you a little dizzy at first, so come back and read it again later. If you're reading this for the first time, you might experience that same feeling you get when you've been on the Tilt – A – Whirl one too many times at the fair. (Never been on the Tilt – A – Whirl? ... well, take my word for it, it's not something you want to ride 10 minutes after eating a corn dog and a funnel cake!) Well, I warned you, but you've kept on reading anyway, so here it goes ...

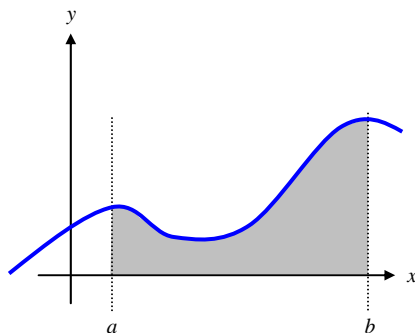
The key to finding the total distance traveled in the last example in a method similar to the first example is to break the time intervals into such short segments, that the velocity over those time segments is almost constant (this will require quite a few intervals). If the velocity is almost constant for each time interval, then we can find the distance traveled for each time interval (which is just the area of an extremely thin rectangle) and add all the areas of all the rectangles together. Sounds simple enough, right? Can you guess what extremely important calculus concept is involved?

We will spend MUCH more time with this later, but it turns out that if we are given the graph of a rate of change (like velocity in miles per hour) we will be able to find the total accumulated change over an interval (like total distance traveled, in mile) by finding the area under the curve.

OK, that last paragraph or two may not have made perfect sense to you ... YET! ... For now, THE GOAL is to figure out a way to find the area under the curve. This chapter actually discusses 5 ways to approximate this area, but we're only going to deal with 4 of them.

**The Area Problem and the Rectangular Approximation Method (RAM)**

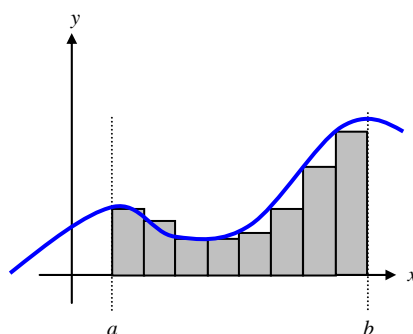
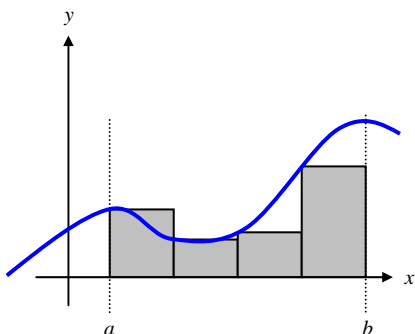
The limit process can be used to find the area under a curve, and we will get into this in more detail in the next section. Suppose we wanted to know the area of the region bounded by a curve, the  $x$ -axis, and the lines  $x = a$  and  $x = b$ , as shown below.



The first step is to divide the interval from  $a$  to  $b$  into subintervals. The examples below show 4 and 8 subintervals, respectively.

After dividing the given interval into subintervals, we can then draw rectangles using the width of each subinterval.

The height of each rectangle is determined by the function value at a point in the specific subinterval, and can be determined using 3 different methods. We could use the left endpoint of each subinterval (called LRAM), the right endpoint of each subinterval (RRAM), or the midpoint of each subinterval (MRAM). Which method is shown below?



*Example:* The total area under the curve then is approximately equal to the total area of all the rectangles. Which of the graphs above gives a better approximation of the area under the curve? Why? How could it be further improved?

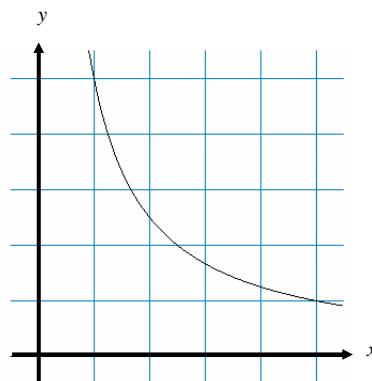
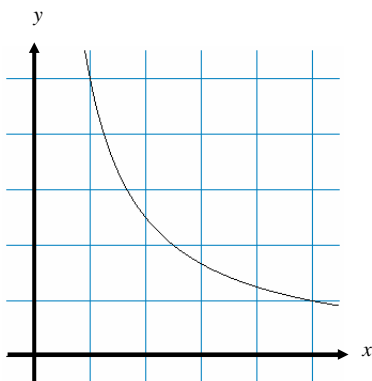
**Summary of the Process:** A sketch is almost mandatory!

Step 1: Divide (or Partition) the interval into  $n$  subintervals.

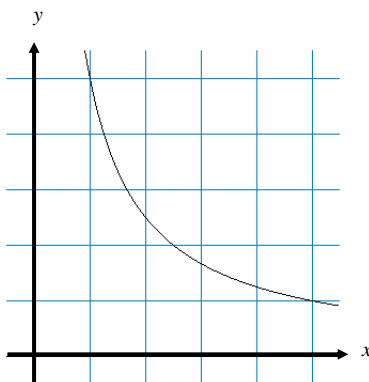
Step 2: Create  $n$  rectangles whose base equals the width of each subinterval and whose height is determined by the function value at the left endpoint, the right endpoint, or the midpoint of the subinterval.

Step 3: Find the area of all  $n$  rectangles and add them together.

*Example:* The graph of  $y = \frac{5}{x}$  is shown twice below. On the left picture approximate the area under the curve from  $x = 1$  to  $x = 5$  using LRAM with 4 rectangles. On the right picture, approximate the area under the curve from  $x = 1$  to  $x = 5$  using RRAM with 4 rectangles. Sketch the rectangles on each curve.



*Example:* Approximate the area under the curve from  $x = 1$  to  $x = 5$  using MRAM with 4 rectangles. Sketch the rectangles on the curve.



*Example:* It is not necessary to have a graph to estimate the area. Suppose the table below shows the velocity of a model train engine moving along a track for 10 seconds. Estimate the distance traveled by the engine, using 10 subintervals of length 1 with (a) left – endpoint values (LRAM) and (b) right – endpoint values (RRAM)

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

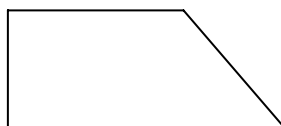
♪: All the examples on this page had a subinterval length equal to 1. This is not always the case, but was done to make the initial examples straightforward. Try doing the first example again using 5 rectangles instead of 4.

**The Trapezoidal Rule (Really §5.5)**

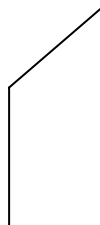
While rectangles make a fairly good approximation, it's easy to see that we're going to need a lot of them to provide a good estimate. We can find a better estimate in less time if we use trapezoids. If we were to partition the interval into subintervals like we did before, we can use each subinterval to create a trapezoid if we just connect the function values of the left and right endpoints. Before we begin, let's make sure you understand the area formula for a trapezoid.

$$\text{Area of a Trapezoid: } A = \frac{1}{2} \cdot h \cdot (b_1 + b_2)$$

While not all trapezoids must look like this, the one's we're going to be using will, so we'll stick with this picture. Label all the parts of the area formula on the picture below.

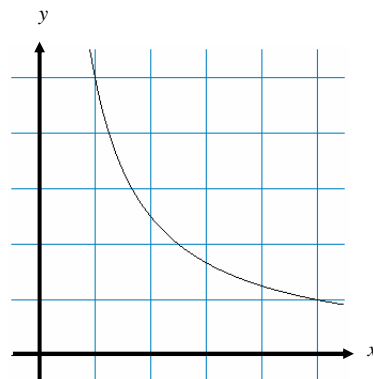


The biggest difference will be the orientation of the trapezoid. The ones we are going to be drawing will look like



Draw a set of axes on the picture above and a function that goes through the top left and top right points of the trapezoid. The "height" of the trapezoid is just the width of a subinterval, and the "bases" are going to be the function values of the left and right endpoints.

*Example:* Let's go back to the same function we used before. Use 4 trapezoids to approximate the area under the curve  $y = \frac{5}{x}$  from  $x = 1$  to  $x = 5$ . Sketch the trapezoids on the curve.



While all we're really doing is finding the area of a bunch of trapezoids, there is always a formula available.

*The Trapezoid Rule*

To approximate the area under a curve on the interval  $[a, b]$  use

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

where  $[a, b]$  is partitioned into  $n$  subintervals of equal length  $h = \frac{(b-a)}{n}$ .

*Proof:*

For all you formula memorizers ... the key words in the above formula is EQUAL LENGTH. It doesn't work so well in the following example:

*Example:* [1998 AP Calculus AB #85 ... with calculator] The function  $f$  is continuous on the closed interval  $[2, 8]$  and has values that are given in the table below.

$x$	2	5	7	8
$f(x)$	10	30	40	20

Using the subintervals  $[2, 5]$ ,  $[5, 7]$ , and  $[7, 8]$ , what is the trapezoidal approximation of the area under the curve?

- A) 110                      B) 130                      C) 160                      D) 190                      E) 210

By the way ... The trapezoid rule connects the left and right hand endpoints with a segment. This method of approximation turns out to be a pretty good, but if you were to connect the endpoints with a curve (namely a parabola) the approximation would be even better. Connecting the endpoints with a parabola and finding the area of the resulting shape is the basis behind the fifth method of approximation called Simpson's Rule. You can read about it on pages 291 – 293 if you find yourself just dying of curiosity, but it's not on the AP exam.