

## 5.2 DEFINITE INTEGRALS

### Riemann Sums

In the last section we found the area under a curve by finding the area of a finite number of rectangles (LRAM, RRAM, and MRAM) or a finite number of trapezoids (TRAPEZOID RULE). Every one of these was an example of what is called a **Riemann Sum**. We're going to stick with RECTANGLES for the time being.

The following steps illustrate what has to happen in order for the sum to be considered a RIEMANN Sum. Read each of the following steps and then illustrate these on the function at the bottom of the page.

Step 1: Start with a *continuous* function on a *closed* interval.

Step 2: **Partition** the interval into  $n$  subintervals. (Previously we made sure they were all the same size, but it turns out, it really doesn't matter for what we're doing here.)

The  $k$ th subinterval will be  $[x_{k-1}, x_k]$  and will have length  $\Delta x_k = x_k - x_{k-1}$

Step 3: In each subinterval, pick **any** number and call the number picked from the  $k$ th subinterval  $c_k$  (for LRAM we picked the left endpoint ... for RRAM we picked the right endpoint ... for MRAM we picked the midpoint)

Step 4: For each interval, using the width,  $\Delta x_k$ , of the interval as the base, create a rectangle that extends from the  $x$ -axis to the function value,  $f(c_k)$ , of the number you picked in each interval.

♫: Some of these rectangles could lie below the  $x$ -axis.

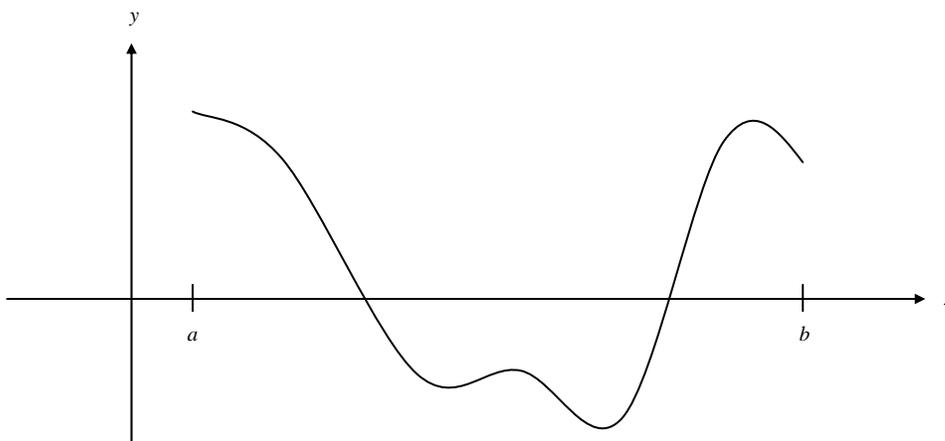
Step 5: On each interval, form the product  $f(c_k) \cdot \Delta x_k$ . (This would have been the area of the rectangles we found previously ... but since any of the rectangles below the  $x$ -axis would have a negative product, we don't call this the area)

Step 6: Find the SUM of each of these products.

In other words find  $f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2 + f(c_3) \cdot \Delta x_3 + f(c_4) \cdot \Delta x_4 + \dots + f(c_n) \cdot \Delta x_n$ .

Another way to write this is  $\sum_{k=1}^n f(c_k) \cdot \Delta x_k$

Following these steps gives you a **Riemann Sum for  $f$  on the interval  $[a, b]$** . Every Riemann sum depends on the partition you choose (i.e. the number of subintervals) and your choice of the number within each interval,  $c_k$ .



*Definite Integral as a Limit of a Riemann Sum*

Goal: Develop a Mathematical DEFINITION of a Definite Integral ... (i.e. What is a Definite Integral)

We used these sums to approximate the *area under the curve*. Again, since area is always positive, we cannot solely use this to describe all Riemann Sums. The same concept applies, however, when we want to find a better approximation.

Using the notation  $P$  to represent the partition, the longest subinterval length would be called the **norm** of  $P$  and is denoted  $\|P\|$ . One of the ways to ensure we have a better approximation is to increase the number of rectangles to infinity, and one of the ways to do this is to make sure that the norm of  $P$  goes to 0.

In order to make the lengths of all the subintervals infinitesimally small, we use a limit as follows:

*Definition of the Definite Integral as a Limit of a Riemann Sum*

Let  $f$  be a function defined on a closed interval  $[a, b]$ . For any partition  $P$  of  $[a, b]$  let the numbers  $c_k$  be chosen arbitrarily in the subintervals  $[x_{k-1}, x_k]$ . If there exists a number  $I$  such that

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \cdot \Delta x_k = I$$

no matter how  $P$  and the  $c_k$ 's are chosen, then  $f$  is **INTEGRABLE** on  $[a, b]$  and  $I$  is the **DEFINITE INTEGRAL** of  $f$  over  $[a, b]$ .

Important  $\mathcal{A}$  (Actually it's a Theorem): IF the function is continuous, THEN the Definite Integral will exist. However, the converse, while true some of the time is NOT ALWAYS true.

Another way to define Definite Integrals is  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \cdot \Delta x$ . Can you explain how this is different than the above definition and what it means for your partition?

*Notation for Definite Integrals*

The limit notation we used last is the form we will use to develop Integral notation. As the number of rectangles goes to infinity, the width of each rectangle,  $\Delta x$ , goes to zero. As we did in the section on differentials, we are going to use the notation  $dx$  to represent this infinitely tiny distance.

The summation notation of sigma is going to be replaced with an *Integral Sign*,  $\int$ , which look somewhat like a giant "S" for sum.

The  $f(c_k)$  which represented a different function value for each interval is going to be replaced with  $f(x)$  since the  $x$ -values are going to be soooooo close together it's almost as if we are evaluating the function at EVERY  $x$ -value in the interval  $[a, b]$ . Combining all of this we have the following notation:

$$\int_a^b f(x) dx$$

We read the notation above as "The Integral of  $f$  of  $x$  from  $a$  to  $b$ "  
Label each part of the notation above.

*Using Definite Integrals as Area*

Ok ... back to the idea of Area. We can define the **area under the curve  $y = f(x)$  from  $a$  to  $b$**  as an *integral* from  $a$  to  $b$  AS LONG AS THE CURVE IS NONNEGATIVE AND INTEGRABLE on the closed interval  $[a, b]$ .

Example: For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then evaluating each definite integral to find it.

a)  $\int_2^9 3 \, dx$

b)  $\int_{-2}^1 |x| \, dx$

c)  $\int_{-3}^3 \sqrt{9-x^2} \, dx$

Example: Consider the function  $f(x) = 3 - x$ . Sketch a graph of this function.

a) What is the "area" between the curve and the  $x$ -axis between  $x = 4$  and  $x = 8$ ?

b) Evaluate  $\int_4^8 (3-x) \, dx$

c) Explain the difference between thinking of a definite integral as area under a curve in the first two examples and this example.

Example: If you knew that  $\int_0^{\pi} \sin x \, dx = 2$ , complete the following using what you know about definite integrals, area, and the sine function.

a)  $\int_{\pi}^{2\pi} \sin x \, dx$

b)  $\int_0^{2\pi} \sin x \, dx$

c)  $\int_0^{\pi/2} \sin x \, dx$

d)  $\int_{-\pi}^{\pi} \sin x \, dx$

e)  $\int_0^{\pi} (2 + \sin x) \, dx$

f)  $\int_0^{\pi} 2 \sin x \, dx$

The FnInt Function of your TI – 83+

By this point, hopefully you understand the following concepts:

1. The limit of a Riemann Sum is used to define a Definite Integral
2. A Definite Integral can be used to find the Area under a curve if the curve is above the  $x$  – axis, and if the curve is below the  $x$  – axis the value of the definite integral is "negative area" ... even though no one in their right mind would ever actually use that phrase in a math class if they wanted to be taken seriously!
3. Since the Definite Integral can be thought of as Area, you can draw a picture and use geometric formulas to find the areas.

BUT ... what happens if you were asked right now, this instant, today to find the definite integral of a function that doesn't lend itself to nice geometric shapes?

The good news for now, is you don't even have to worry about how to do these by hand! You get to use your calculator!

#### How to Use Your Calculator to Find a Definite Integral:

The syntax for using your calculator is as follows:  $\text{fnInt}(\text{function}, x, \text{lower bound}, \text{upper bound})$

1. Press **MATH**
2. Press 9: **fnInt**(
3. Enter the function followed by **,**
4. Press **[X,T,Θ,η]** followed by **,**
5. Enter the lower bound followed by **,**
6. Enter the upper bound
7. Close the parenthesis

*Example:* Evaluate  $\int_1^5 \frac{5}{x} dx$  ... if you have your notes from the last section, go back and compare this answer to your answers for LRAM, RRAM, MRAM, and the Trapezoid Rule.

*Example:* Evaluate  $3 + 2 \int_0^{\pi/2} \tan x dx$

You can also do the same thing from the graphing screen.

*Example:* Graph  $y = \sqrt{x}$  on a standard viewing window. Evaluate  $\int_1^8 \sqrt{x} dx$ .

Press **2nd****TRACE** (CALC), 7:  $\int f(x)dx$ , enter Lower Bound as 1, enter Upper Bound as 8.

Compare this to using to the previous method.