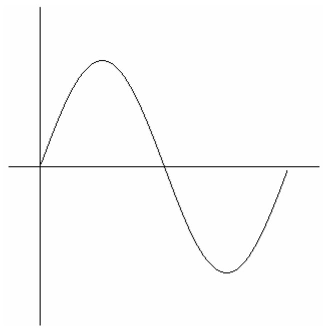
5.4 FUNDAMENTAL THEOREM OF CALCULUS

Do you remember the Fundamental Theorem of Algebra?

The Fundamental Theorem of Calculus has two parts. These two parts tie together the concept of integration and differentiation and is regarded by some to by the most important computational discovery in the history of mathematics!

In order to start developing this concept, we are going to use what I will call an "Area Accumulation Function".

Example: The graph of $f(t)$ given below has odd symmetry around the point (2, 0). On the interval [0, 2], the graph is symmetric with respect to the line $t = 1$. Also, $\int_0^1 f(t)\,dt = \frac{4}{3}$.



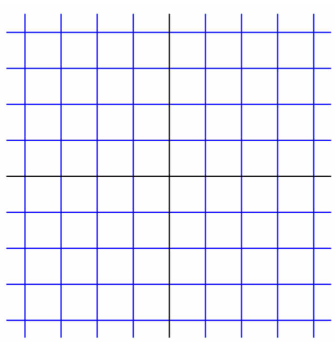
Example: Let $F(x) = \int_0^x f(t)\,dt$.

a) Complete the following table:

x	0	1	2	3	4
$F(x)$					

b) Sketch your best estimate of the graph of F on the grid below. (This is an "Area Accumulation Function")

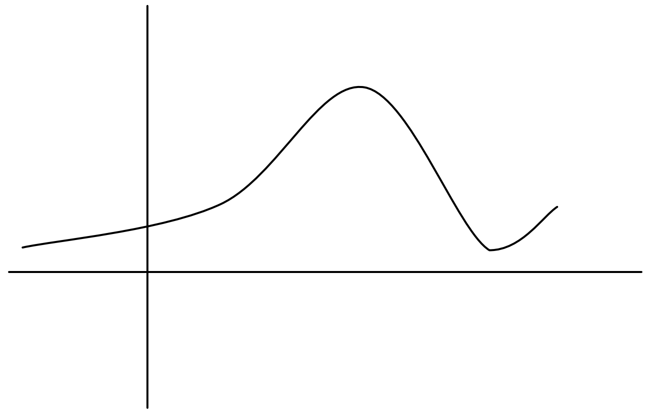
$$y = F(x)$$



Now that we've established what basically an "Area Accumulation Function" is, let's change the notation slightly just for this next part.

We're going to use ____________________ to describe the Area Accumulation Function from *a* to *x*.

Consider the graph below to be $f(t)$. We are interested in the change in Area from $t = x$ to $t = x + h$.



Example: $A_a^x + A_x^{x+h} =$

Example: Solve the expression above for A_x^{x+h} :

Example: Does it really matter what value of *a* we use? The last expression can be written as ____________________.

In the last section, we talked about the properties of definite integrals. There was one we did not discuss, and we need it now.

#6. *Max-Min Inequality*: If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$

Example: Using this property with our Area notation, we can bound A_x^{x+h} using

Example: Divide all three parts of the inequality above by *h*. What can be said then about $\dfrac{A_x^{x+h}}{h}$ as *h* goes to 0?

Hint: Remember the Sandwich (Squeeze) Theorem?

Example: Since the Sandwich Theorem tells us that $\lim\limits_{h\to 0}\frac{A_x^{x+h}}{h} = f(x)$, replace A_x^{x+h} with the expression we used in a previous example.

Example: What is the left side of the equation above equal to?

What have we just done? We have said that the *derivative of the area accumulation function under* $f(x)$ is equal to $f(x)$. Instead of using the A notation to describe this, we could have used an integral.

The Fundamental Theorem of Calculus [Part #1 … Simple]

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

In other words, the Integral and the Derivative are just __________________________.

Example: $\frac{d}{dx}\left[\int_3^x \left(5t^2-6t+1\right)dt\right]$

Now, for part two!

Example: If we know that $A'(x) = f(x)$, then what does $A(x) =$

Example: How do we find C ?

Since $A(x)$ was are "area accumulation function" we can use an integral to represent $A(x)$ to obtain the second part of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus [Part 2: The Evaluation Part]

If f is continuous at every point of $[a, b]$,

$$\int_a^x f(x)\,dx = F(x) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

Example: $\int_0^3 x^2\ dx$

And now, for the last part … Part 1 Extended!

Example: Using the second part of the Fundamental Theorem of Calculus, $\int_a^x f(x)\,dx = F(x) - F(a)$, take the derivative of both sides. The result should be familiar.

What if *a* was not a constant and *x* was more complicated. In other words, what if the limits of integration were themselves functions of *x*?

Example: $\int_{v(x)}^{u(x)} f(t)\,dt =$

Example: Take the derivative of both sides of the equation above. Remember, $F'(x) = f(x)$.

We now have the extended version of the first part of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus [Part 1 … Extended]

$$\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)\,dt\right] = f\left(u(x)\right)\cdot u'(x) - f\left(v(x)\right)\cdot v'(x)$$

Example: