6.3 INTEGRATION BY PARTS

When you integrate, you find an antiderivative, or you "undo" the derivative. When you used the substitution method, you were undoing a derivative that involved the Chain Rule. In this section we will learn how to undo a derivative that involved the Product Rule.

First, recall the way we differentiate functions using the Product Rule and differential notation.

$$(uv)' = u\ dv + v\ du$$

Using the Product Rule above, we will develop a "formula" for Integration by parts.

Begin by taking the Integral of both sides.

Then rearrange the equation, solving for $\int u\ dv$.

♬: You could have just as easily solved for $\int v\ du$, but it seems like every Calculus book uses the first way!

Integration by Parts

If *u* and *v* are functions of *x* and have continuous derivatives, then

$$\int u\ dv = uv - \int v\ du$$

Here's what you do … In order to solve integrals of this type, you need to decide what to let *u* and *dv* equal. Then follow the formula in the box above. **In order to find *v* from *dv* you must integrate, so choose wisely**.

Guidelines to Choose *u* and *dv*

- Let *dv* be the part that you are able to integrate.
- It helps if *du* is simpler than *u* (or at least no more complicated).
- It helps if *v* is simpler than *dv* (or at least no more complicated).

L I P E T

Further guidelines to choosing *u*:

First choice is a natural logarithm … *L* ….
Second choice is an inverse trigonometric function … *I* …
Third choice is a polynomial … *P* …
Fourth choice is an exponential … *E* …
Lastly, chose a trigonometric function … *T* …

Example: $\int xe^x \, dx$

Example: $\int \theta \sec\theta \tan\theta \, d\theta$

Example: $\int \frac{\ln x}{x^2} \, dx$

Example: $\int x^2 \ln x \, dx$

Special Cases of Integration by Parts

Example: $\int \arccos x \, dx$

Example: $\int \ln x \, dx$

Integrals of Trigonometric Functions

First, a few identities from trigonometry that you may or may not remember.

1. $1+\tan^2 x = \sec^2 x$ … everyone remembers this one, right?!

2. $\sin^2 x = \dfrac{1-\cos(2x)}{2}$ … This is called a Power Reducing Formula or the Half – angle identity for $\sin^2 x$, and if you remember this one, you probably don't need me to help you with the rest of this course ☺

3. $\cos^2 x = \dfrac{1+\cos(2x)}{2}$ … Same as #2 … It's a Power Reducing Formula or the Half – angle identity for $\cos^2 x$.

Can you integrate all of these functions? The first 4 should already be known.

$\int \sin x\, dx$ $\int \cos x\, dx$ $\int \sec^2 x\, dx$ $\int \csc^2 x\, dx$

$\int \tan x\, dx$ $\int \cot x\, dx$

$\int \csc x\, dx$ $\int \sec x\, dx$

$\int \sin^2 x\, dx$ $\int \cos^2 x\, dx$

$\int \tan^2 x\, dx$ $\int \cot^2 x\, dx$

Repeated Use of Integration by Parts

There are two ways to repeatedly use integration by parts. The first way just involves a repeated application of integration by parts until you reach an integral you can finish without integration by parts. The second involves using integration by parts and collecting like integrals on one side.

Example: $\int x^2 \cos x \, dx$

Example: $\int e^{2x} \sin x \, dx$

Tabular Method

In problems involving repeated applications of integration by parts, a tabular method can help to organize the work. Keep in mind, you are still using integration by parts repeatedly, but you are using the table to organize the repetitions. This method works well for integrals of the form $\int x^n \sin(ax)\, dx$, $\int x^n \cos(ax)\, dx$, $\int x^n e^{ax}\, dx$.

Example: $\int x^3 \sin(4x)\, dx$

Definite Integration with Integration by Parts

Example: $\int_0^{\pi/2} x \cos x \, dx$

Algebraic Techniques

Long Division

Example: $\int \frac{x^2-1}{x^2+1}\,dx$

Expand

Example: $\int (\sin x+\cos x)^2\ dx$

Complete the Square:

Example: $\int \frac{2\,dx}{x^2-6x+10}$

Example: $\int_0^1 \frac{3\,dx}{(x+1)\sqrt{x^2+2x}}$

Example: $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

Separate the numerator

Example: $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$

Example: $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$