4.1 EXTREME VALUES OF FUNCTIONS

Extrema (plural for extremum) are the maximum or minimum values of functions. We need to distinguish between absolute extrema and relative extrema, and how to locate them. We have already looked at quadratic functions and you have used your calculator to find the extrema in the past.

Definition of Absolute Extrema

Let f be defined on an interval I containing c.

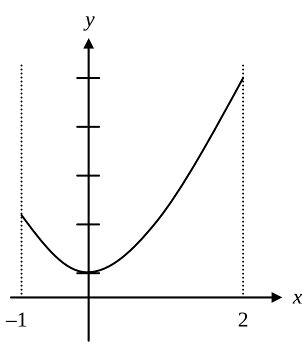
1. $f(c)$ is the **minimum of *f* on the interval *I*** if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of *f* on the interval *I*** if $f(c) \ge f(x)$ for all x in I.

The Extreme Value Theorem

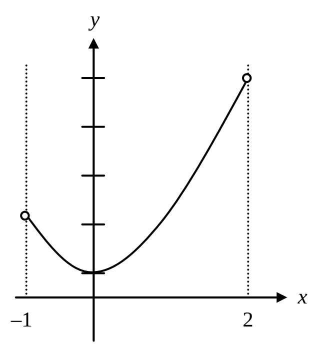
If f is continuous on a closed interval [a, b], then f has both a minimum and a maximum on the interval

Example: Using the graphs provided, find the minimum and maximum values on the given interval. If there is no maximum or minimum value, explain which part of the Extreme Value Theorem is not satisfied.

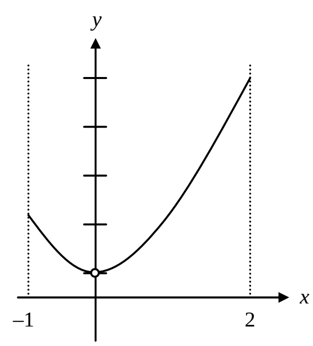
(a) [–1, 2]



(b) (–1, 2)



(c) [–1, 2]



Example: Draw a sketch to find the absolute extrema of the function $f(x) = \sqrt{4 - x^2}$ on the interval

(a) [–2, 2]

(b) [–2, 0)

(c) (–2, 2)

(d) [1, 2)

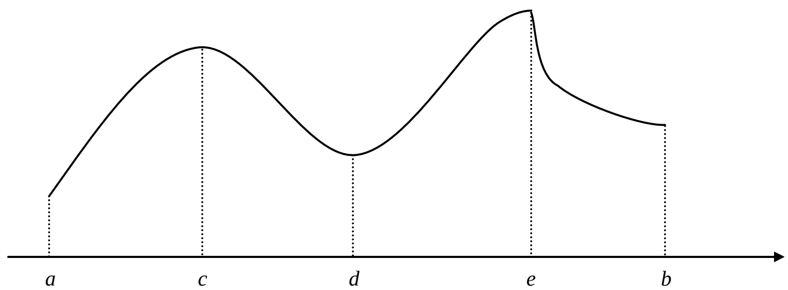
Relative Extrema and Critical Numbers

Definition of Relative Extrema

1. If there is an open interval containing *c* on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum**.
2. If there is an open interval containing *c* on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum**.

Basically relative extrema exist when the value of the function is larger (or smaller) than all other function values relatively close to that value.

Example: The maximum and minimum points in the last example occurred at either the endpoints or at points interior to the interval. Suppose our function looked like the graph below. Label the points *a* – *e* as absolute or relative extrema.



When given a graph it is fairly simple to identify the extrema. The question to be asked then is how do we find the extrema when we do not have a graph given to us?

Example: Except at the endpoints *a* and *b*, what do you notice about the derivative at the relative extrema in the last example?

Definition of a Critical Point

Let *f* be defined at *c*. If $f'(c) = 0$ <u>or</u> if f' is undefined at *c*, then *c* is a **critical point** of *f*.

Relative Extrema Occur Only at Critical Points

If *f* has a relative minimum or relative maximum at $x = c$, then *c* is a critical number of *f*.

IMPORTANT ♬: Just because the derivative is equal to zero (or undefined) does not mean there is a relative maximum or minimum there. *Relative extrema* can occur ONLY at critical points, and critical points occur ONLY when the derivative is either 0 or undefined, however, it is possible for the derivative to equal zero (or undefined) and there be NO extrema there. If you need to convince yourself of this, try $f(x) = x^3$ and $f(x) = x^{1/3}$ at $x = 0$.

Guidelines for Finding Absolute/Relative Extrema on a Closed Interval

1. Find the critical numbers of *f* in (*a*, *b*). Do this by ____.

2. Evaluate *f* at each critical number in (*a*, *b*).

3. Evaluate *f* at the endpoints of [*a*, *b*].

4. The least of the values from steps 2 and 3 is the minimum, and the greatest of these values is the maximum. All the other values are relative extrema. For now, a quick check of the values on either side of the critical number will help determine whether or not the function value at that critical number is a maximum or a minimum.

Example: The critical numbers (or critical points) are _______ values, while the maximums/minimums of the function are _______ values.

Example: Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval [–1, 2]. Use your graphing calculator to investigate first.

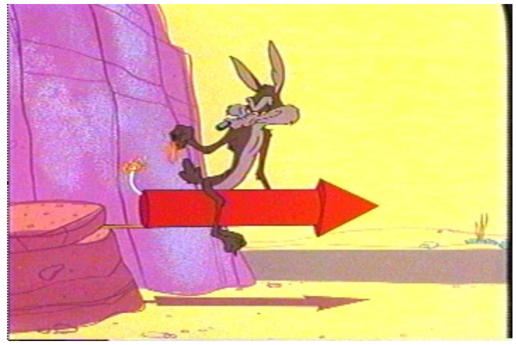
Example: Find the extrema of $g(x) = 2x - 3x^{2/3}$ on the interval [–1, 3]. Use your graphing calculator to investigate first.

Example: Find the extrema of $h(\theta) = 2\sin\theta - \cos(2\theta)$ for $0 \le \theta \le 2\pi$. Use your graphing calculator to investigate first.

Example: Wile E. is after Road Runner again! This time he's got it figured out. Sitting on his ACME rocket he hides behind a hill anxiously awaiting the arrival that "beeping" bird. In his excitement to light the rocket he tips the rocket up. Instead of thrusting himself parallel to the ground where he can catch the Road Runner, he sends himself widely into the air following a path given by the position function

$$s(t) = -.00086x^4 + .067x^3 - 1.67x^2 + 14.77x.$$

When does Wile E. Reach his maximum height?



Optimization … really §4.4

The concepts of finding the maximum or minimum of a function lay the foundation for the mathematical theory behind optimization. To *optimize* something means to maximize or minimize some part of it. Remember, *if you have a closed interval you MUST test the endpoints of the function as well.*

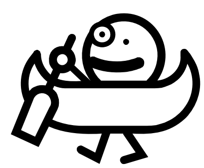
In every optimization problem, your goal is to find the maximum or minimum value of a function representing some real-world quantity. Your first (and often the hardest) task is always to find an expression for the function to be optimized. This involves translating the problem into mathematical terms. Once you have the function, you can then apply the methods we have learned to determine the maximum or minimum value.

Steps for Solving Applied Optimization Problems

Step 1: Understand the problem. Read it carefully, and ask yourself, "Self, what is the quantity to be maximized or minimized? What are the quantities which it depends on?"
Step 2: Draw a diagram if possible. Sometimes, more than one diagram helps you to determine how all the quantities are related. Identify all quantities from step 1 on your diagram.
Step 3: Assign variables to the quantities from step 1.
Step 4: Determine a function for the quantity to be optimized.
Step 5: Eliminate all but ONE variable and determine the domain of the resulting function. This usually requires a second equation.
Step 6: Optimize the function.
(a) Calculate the derivative and find the critical numbers
(b) If the domain is a closed interval, compare the function's value of the critical numbers with that of the endpoints.
(c) If the domain is an open interval (or infinite interval), use the first or second derivative tests to analyze the behavior of the function. (This will be explained in the next few sections)

Example: Find two nonnegative real numbers that add up to 66 and such that their product is as large as possible.

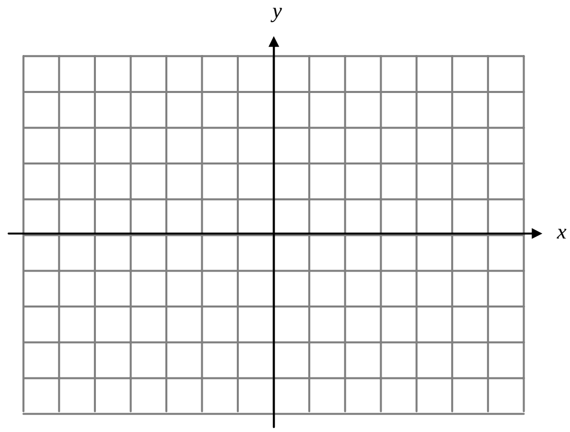
Example: You are in a rowboat on Lake Erie, 2 miles from a straight shoreline taking your potential in-laws for a boat ride. Six miles down the shoreline from the nearest point on shore is an outhouse. You suddenly feel the need for its use. It is October, so the water is too cold to go in, and besides, your in-laws are already pretty unimpressed with your "yacht". It wouldn't help matters to jump over the side and relieve your distended bladder. Also, the shoreline is populated with lots of houses, all owned by people who already have restraining orders against you. If you can row at 2 mph and run at 6 mph (you can run faster when you don't have to keep your knees together), for what point along the shoreline should you aim in order to minimize the amount of time it will take you to get to the outhouse? (…And you thought calculus wasn't useful!)



4.2 MEAN VALUE THEOREM

The Mean Value Theorem is considered by some to be the most important theorem in all of calculus. It is used to prove many of the theorems in calculus that we use in this course as well as further studies into calculus.

Example: Graph the points (–6, 4) and (5, –4) on the grid below.



Example: Draw a non – linear function, passing through the points above, that is continuous on the closed interval [–6, 5] and differentiable on the open interval (–6, 5).

Example: Draw a line between the points (–6, 4) and (5, –4). Calculate the slope of this line.

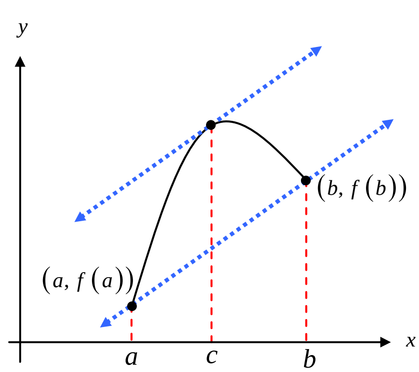
Example: Are there any other points on your function, where the tangent line has the same slope as the line joining (–6, 4) and (5, –4)? Sketch these tangent lines.

The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then ***there exists*** a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Just like the Intermediate Value Theorem, this is an *existence theorem*. The Mean Value Theorem does not tell you what the value of c is, nor does it tell you have many exist. Again, just like the Intermediate Value Theorem, you must keep in mind that c is an x – value.



Also, the hypothesis of the Mean Value Theorem (MVT) is highly important. If any part of the hypothesis does not hold, the theorem cannot be applied.

Basically, the Mean Value Theorem says, that the average rate of change over the entire interval is equal to the instantaneous rate of change at some point in the interval.

Example: A plane begins its takeoff at 2:00 pm on a 2500-mile flight. The plane arrives at its destination at 7:30 pm (ignore time zone changes). Explain why there were at least two times during the flight when the speed of the plane was 400 miles per hour.

Example: Apply the Mean Value Theorem to the function on the indicated interval. In each case, make sure the hypothesis is true, then find all values of *c* in the interval that are guaranteed by the MVT.

a) $f(x) = x\left(x^2 - x - 2\right)$ on the interval [–1, 1]

b) $f(x) = \dfrac{x+1}{x}$ on the interval [0.5, 2].

c) $f(x) = -2x^2 + 14x - 12$ on the interval [1, 6]

The last example is a special version of the Mean Value Theorem called Rolle's Theorem. In fact, the proof of the Mean Value Theorem can be done quite easily, if you prove Rolle's Theorem first. Rolle's Theorem basically states that if the function is continuous on the closed interval and differentiable on the open interval AND the values of the function at the endpoints are equal, then there must exist at least one point in the interval where the derivative is zero.

Consequences of the Mean Value Theorem

While the Mean Value Theorem is used to prove a wide variety of theorems, we will be focusing on the results and/or consequences of the Mean Value Theorem. In this section, we will discuss when a function increases and decreases as well as a brief introduction to antiderivatives.

Increasing versus Decreasing

Why mathematicians feel the need to define everything is a mystery you will probably never figure out unless you become one. Then, for some inexplicable reason, you will find yourself questioning the truthfulness of every argument ever made, reducing every argument to its basic foundational vocabulary, and finally analyzing the very soul and fiber of the definitions. With this in mind, we will now define what it means for a function to be increasing and decreasing. (Obviously, we cannot just say that a function is increasing when all the function values get bigger.)

Definitions of Increasing and Decreasing Functions

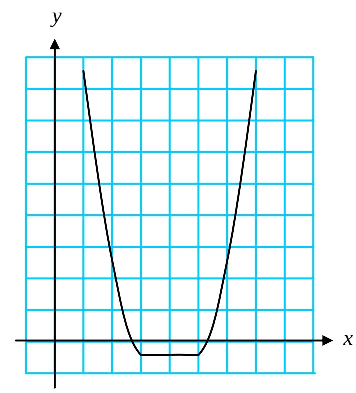
A function *f* is **increasing** on an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function *f* is **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

Example: What interval is the function decreasing? increasing? constant?



Example: What is the value of the derivative when the function is decreasing? increasing? constant?

Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval [a, b] and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ <u>for all</u> x in (a, b), then f is increasing on [a, b].
2. If $f'(x) < 0$ <u>for all</u> x in (a, b), then f is decreasing on [a, b].
3. If $f'(x) = 0$ <u>for all</u> x in (a, b), then f is constant on [a, b].

The proof of the above concepts comes directly from the Mean Value Theorem.

Guidelines for Finding Intervals on Which a Function is Increasing or Decreasing

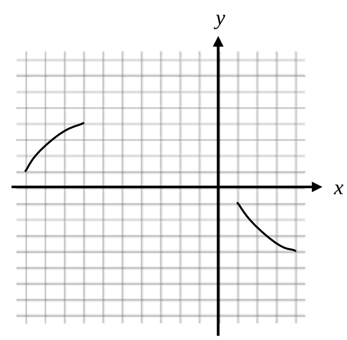
Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing use the following steps:

1. Find the <u>critical points</u> of f in the interval (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at ONE test value in each interval.
3. Use the <u>signs of the derivative</u> to determine whether or not the function is increasing or decreasing.

Example: Find the intervals on which $f(x) = x^3 - 12x - 5$ is increasing or decreasing.

Example: Find the intervals on which $f(x) = \left(x^2 - 9\right)^{2/3}$ is increasing or decreasing.

Example: The figure to the right gives two parts of the graph of a continuous differentiable function f on [–10, 4]. The derivative f' is also continuous.

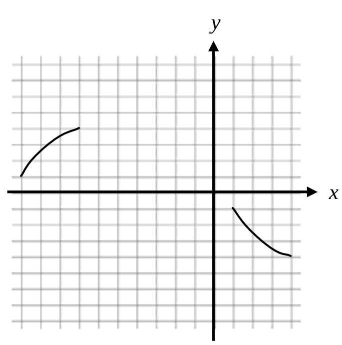


a) Explain why f must have at least one zero in [–10, 4].

b) Explain why f' must also have at least one zero in the interval [–10, 4]. What are these zeros called?

c) Make a possible sketch of the function with one zero of f' on the interval [–10, 4].

d) Make a possible sketch (on the graph below) of the function with <u>two</u> zeros of f' on the interval [–10, 4].

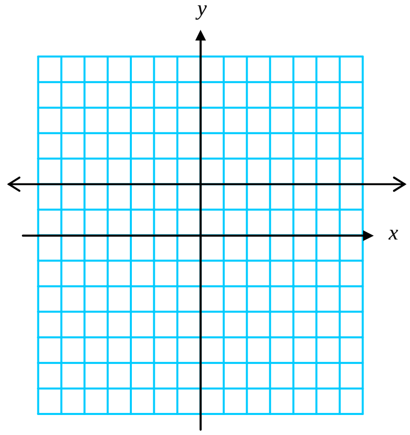


Antiderivatives

Example: Suppose you were told that $f'(x)=2x-1$. What could $f(x)$ possibly be? Is there more than one answer?

Finding the function from the derivative is a process called ***antidifferentiation***, or finding the antiderivative.

Example: Suppose the graph of $f'(x)$ is given below. Draw at least three possible functions for $f(x)$.



The three functions you drew should only differ by a constant. If you let C represent this constant, then you can represent the *family* of all antiderivatives of $f'(x)$ to be $f(x)=2x+C$.

Example: If you were told that $f(3)=-2$, what would the value of C be?

IMPORTANT ♬: If a function has one antiderivative it has many antiderivatives that all differ by a constant. Unless you know something about the original function, you cannot determine the exact value of that constant, but it must be in your answer!

Example: If you know that the acceleration of gravity is $-32\,{}^{ft}\!/_{s^2}$, for an falling object, we could write the acceleration of the object at time t as $a(t)=-32$. Find a function for the velocity of the object at time t. What does the constant equal (in words)?

Example: Find a function for the position of the object at time t. What does the constant equal (in words)?

Example: [2004 AP Calculus AB Free Response Question #3 … Calculator Allowed] A particle moves along the y-axis so that its velocity at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}(e^t)$. At time $t = 0$, the particle is at $y = -1$.
(Note: $\tan^{-1} x = \arctan x$)

a) Find the acceleration of the particle at time $t = 2$.

b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

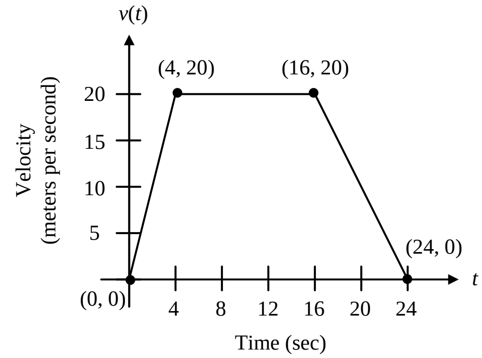
d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

[You can't actually do part d yet … but you should be able to do the following:]

i) Explain what you would need to do in order to find the position at time t = 2 and what information in the original problem is useful (and necessary) for this part of the question.

ii) If you were able to determine position of the particle at t = 2, then you should be able to explain how to determine the direction of the particle at t = 2. For instance, if you knew the position of the particle at t = 2 was –1.361, then how would you determine whether or not the particle was moving toward the origin or away from the origin?

Example: [2005 AB Calculus AP Free Response #5 … No Calculator Allowed] A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity *v*(*t*), in meters per second, is modeled by the piecewise-linear function defined by the graph below.



a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

[Obviously we haven't used this symbol yet, nor have we talked about how to get it … so here's a couple of hints …]

i) If I told you the notation $\int_0^{24} v(t)\,dt$ *only asked you to find the antiderivative of the velocity function, you should be able to use correct units.*

ii) If I told you that all the notation $\int_0^{24} v(t)\,dt$ *means for this problem is to find the area under the given curve, you should then be able to answer the question AND explain the meaning of* $\int_0^{24} v(t)\,dt$.

b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

c) Let *a*(*t*) be the car's acceleration at time *t*, in meters per second per second. For 0 < *t* < 24, write a piecewise-defined function for *a*(*t*).

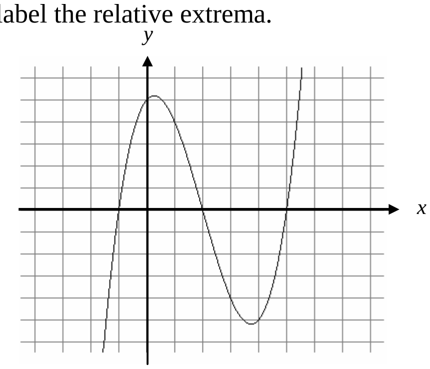
d) Find the average rate of change of *v* over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of *c*, for 8 < *c* < 20, such that $v'(c)$ is equal to this average rate of change? Why or why not?

4.3 CONNECTING $f\,'$ AND $f\,''$ WITH THE GRAPH OF f

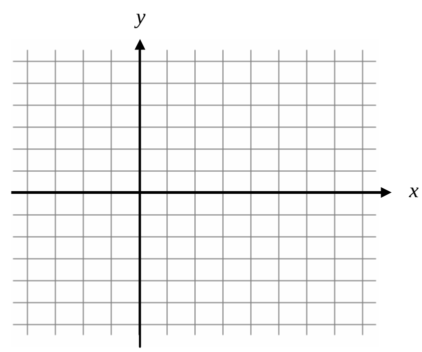
First Derivative Test for Extrema

We have already determined that relative extrema occur at critical points. The behavior of the first derivative before and after those critical points will help determine whether or not the function has a relative maximum or minimum (or neither) at these critical points.

Example: Given the graph of f below, label the relative extrema.



Example: Sketch a graph $f\,'$, on the axis below. Label the x – values of $f\,'$ where the extrema occur on f.



The First Derivative Test

Let f be a continuous function, and let c be a critical point.

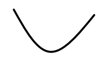
1. If $f\,'$ changes sign from positive to negative at c, then f has a local maximum value at c.
2. If $f\,'$ changes sign from negative to positive at c, then f has a local minimum value at c.
3. If $f\,'$ DOES NOT change signs, then there is no local extreme value at c.

Important ♬: If you are asked to find the absolute maximum (or just a maximum) of a function on a closed interval, you MUST test the endpoints also.

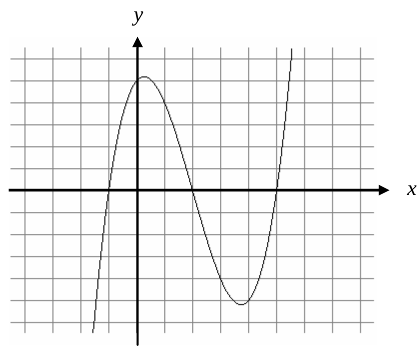
Example: Find where the function $h(x) = x\sqrt{4 - x^2}$ is increasing and decreasing, then use the first derivative test to determine any local extrema.

Example: Find where the function $g(x) = x^2e^x$ is increasing and decreasing, then find any local extrema and absolute extrema.

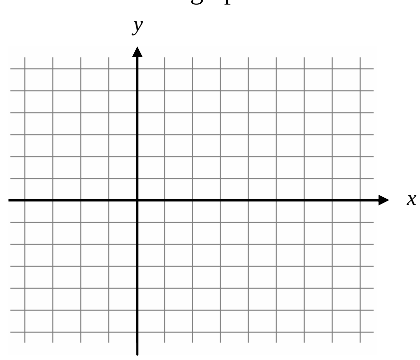
Concavity

Concavity deals with how a graph is curved. A graph that is concave up looks like  , while a graph that is concave down looks like  . We can use the SECOND derivative to determine the concavity of a function.

Example: Using the same graph as our previous example, indicate which portions of the graph are concave up, and which portions are concave down. Label the point where the graph changes concavity, and then sketch the graph of the first derivative <u>on the same graph.</u>



Example: Sketch the graph of the *second* derivative on the graph below. Label the *x*-value where the graph changes concavity.



Example: Using the graphs you sketched, circle the correct word that completes the following statements:

a) When the graph of *f* is increasing, *f* ' is (positive , negative).

b) When the graph of *f* is decreasing, *f* ' is (positive , negative).

c) The graph of *f* is concave upward when *f* " is (positive , negative).

d) The graph of *f* is concave downward when *f* " is (positive , negative).

e) The graph of *f* is concave upward when *f* ' is (increasing , decreasing).

f) The graph of *f* is concave downward when *f* ' is (increasing , decreasing).

Using the results from the last few statements leads us to the following definition.

Definition of Concavity

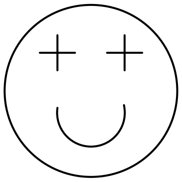
Let $y = f(x)$ be a differentiable function on an interval I. The graph of $f(x)$ is **concave up** on I if f' is increasing on I, and **concave down** on I if f' is decreasing on I.

If the first derivative is increasing, then the second derivative must be __________________. If the first derivative is decreasing, then the second derivative must be __________________. Thus instead of using the definition of concavity to determine whether the function is concave up or down, we can use the following test.

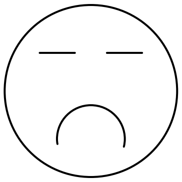
Concavity TEST

The graph of a twice-differentiable function $y = f(x)$ is **concave up** on any interval where $y'' > 0$, and **concave down** on any interval where $y'' < 0$.

The Concavity Test can be summed up by the following pictures … While this is a humorous (and hopefully helpful) way to remember concavity, please understand that this is NEVER to be used as a justification on ANY test!



f'' positive $\Rightarrow$ Concave UP



f'' negative $\Rightarrow$ Concave DOWN

Example: Find the intervals where the function $g(x) = -2x^3 + 6x^2 - 3$ is concave up and concave down.

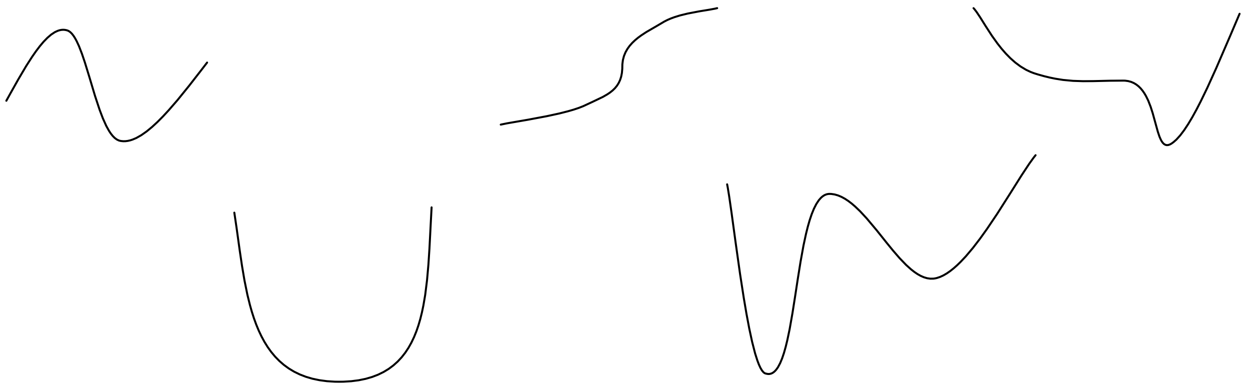
Example: Determine the concavity of $h(x) = x\sqrt{4 - x^2}$.

Points of Inflection

Definition

A point where the graph of a function has a tangent line (even if it's a vertical tangent line) AND where the concavity changes is a **point of inflection**.

Example: Using each picture, estimate each point of inflection, if any, and sketch the tangent line at that point.



Since points of inflection occur when the graph changes ________________, and a graph changes concavity when the ___________________________________ changes from positive to negative (or vice-versa), then if we wanted to find the points of inflection of a graph, we only need to focus on when the second derivative equals 0 (or does not exist)

IMPORTANT ♬: Just because the second derivative equals zero (or does not exist) you are NOT guaranteed that the function has a point of inflection. The second derivative MUST change signs (meaning concavity changed) in order for a point of inflection to exist!

Example: Go back to page 69 of your notes and look at the graph you drew of the second derivative. Estimate the *x*-value of the point of inflection on the original function. Where SHOULD this value be graphed when graphing the second derivative? Fix the graph if necessary.

Example: Find the points of inflection of $g(x) = -2x^3 + 6x^2 - 3$.

Example: Find the points of inflection of $h(x) = x\sqrt{4 - x^2}$.

Second Derivative Test for Extrema

Example: Go back to the original function on page 68. First look at the point where the function had a maximum. Was the graph concave up or down at that point? Secondly, look at the point where the function had a minimum. Was the graph concave up or down at that point?

As long as the function is twice-differentiable (meaning the first derivative is a smooth curve), then we can actually determine whether or not a critical point is a relative maximum or minimum WITHOUT testing values to the right and left of the point. We can use the Second Derivative Test.

Second Derivative Test for Local Extrema

If $f'(c)=0$ (which makes $x = c$ a critical point) AND $\boldsymbol{f''(c)<0}$, then f has a local MAXIMUM at $x = c$.
If $f'(c)=0$ (which makes $x = c$ a critical point) AND $\boldsymbol{f''(c)<0}$, then f has a local MINIMUM at $x = c$.

Important ♬: If the second derivative is equal to zero (or undefined) then the Second Derivative Test is INCONCLUSIVE.

Remember the happy (and sad) faces? If a critical point happens to occur in an interval where the graph of the function is CONCAVE UP, then that critical point is a relative MINIMUM. If a critical point happens to occur in an interval where the graph of the function is CONCAVE DOWN, then that critical point is a relative MAXIMUM.

Example: Use the Second Derivative Test to identify any relative extrema for the function $g(x)=-x^4+4x^3-4x+1$.

NOTES FOR OPTIMIZATION PROBLEMS:

Whenever you are required to Maximize or Minimize a function, you MUST justify whether or not your answer is actually a maximum or a minimum. You may use the FIRST DERIVATIVE TEST (testing points to the left and right of the critical points in the first derivative to see if the sign of the first derivative changes from positive to negative or vice-versa), or the SECOND DERIVATIVE TEST (plugging in the critical points to the second derivative to see if the critical points occur when the original function was concave up or down).

ALWAYS REMEMBER that both of these tests are checking for relative extrema. If you have a CLOSED interval, you must check the endpoints to make sure the absolute maximum or minimum values do not happen to occur there.

Example: [2005 AP Calculus AB Free Response #4 … No Calculator Allowed]

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	–1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	–3	Negative
$f''(x)$	–2	Negative	0	Positive	DNE	Negative	0	Positive

Let f be a function that is continuous on the interval [0, 4]. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exist at $x = 2$.

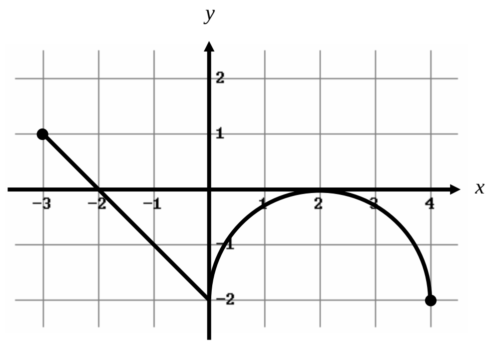
a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

b) On the axes provided, sketch the graph of a function that has all the characteristics of f.

~~c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$ on the open interval (0, 4). For $0 < x < 4$, find all values of x at which g has a relative extremum. Determine whether g has a relative maximum or a relative minimum at each of these values. Justify your answer.~~

~~d) For the function g defined in part c, find all values of x, for $0 < x < 4$, at which the graph of g has a point of inflection. Justify your answer.~~

Example: [2003 AP Calculus AB Free Response #4 … No Calculator Allowed] Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown below.



Graph of f'

a) On what intervals, if any, is f increasing? Justify your answer.

b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

c) Find an equation for the line tangent to the graph of f at the point (0, 3).

~~d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.~~

4.5 LINEARIZATION AND NEWTON'S METHOD

Linearization

The goal of linearization is to approximate a curve with a line. Why? Because it's easier to use a line than a curve! The basic idea of linearization is to find the equation of the tangent line at a certain point, and use the tangent line to estimate the desired value of the original function.

Example: Consider $f(x) = \sqrt{x}$. We all know that $f(4) = 2$, but without a calculator, what is $f(4.1)$?

a) Find the equation of the tangent line for $f(x)$ when $x = 4$, call it $L(x)$.

b) We say that $L(x)$ is approximately the same as $f(x)$ centered at $x = 4$. Use $L(x)$ to approximate $f(4.1)$.

c) Use a calculator to approximate $f(4.1)$. Are you close?

Example: [1969 Multiple Choice AB #36] The approximate value of $y = \sqrt{4 + \sin x}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$, is

A) 2.00 B) 2.03 C) 2.06 D) 2.12 E) 2.24

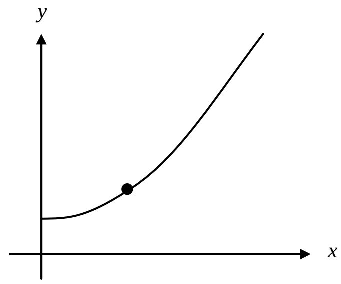
Example: [1973 Multiple Choice AB #44] For small values of h, the function $\sqrt[4]{16 + h}$ is best approximated by which of the following?

A) $4 + \frac{h}{32}$ B) $2 + \frac{h}{32}$ C) $\frac{h}{32}$ D) $4 - \frac{h}{32}$ E) $2 - \frac{h}{32}$

Differentials

Approximations aren't exact! (Aren't you glad you woke up this morning to hear that enlightening bit of information?!) If we use a line to approximate a curve, it gives us a good estimate, as long as we don't go too far away from the center point. Wouldn't it be nice if we knew how far off our approximation is going to be? Well, whether you are excited about this or not, here we go!

Example: Consider the function f below. Label the point $(c, f(c))$, and draw the tangent line at that point.



Example: What is the equation of the *tangent line* you drew?

Move a "small" distance to the right of c. Normally, we would call this distance Δx, but when Δx is very small, we will instead use the notation dx, the **differential of *x***.

Example: What is the **function** value at this point?

Example: What is the value of this point on the *tangent line* (when $x = c + dx$) ?

Example: How much did the y-values ACTUALLY change?
[Find Δy on the **function**]

Example: How much did the y-values APPROXIMATELY change?
[Find dy (the differential of y … a small change in y} on the *tangent line*]

In other words, if we were to use any value of x, the approximate change in y after a small change in x would be written

$$dy = f'(x)dx$$

This should look VERY familiar … What differentials allow us to do is to say that if the ratio of the differentials exists, it will be equal to the derivative. It allows us to write $\frac{dy}{dx}$ as the derivative of y with respect to x, but use the dy and dx as separate terms. Finding a differential is very similar to finding a derivative.

Example: Find the differential dy if $y = x^3 - 5x$

Example: Find $d(\cos 5x)$.

Since dy is the approximate change in the y values when x is changed a small amount, we can use differentials to estimate the change in other problems if we know the small change in x.

Example: Find the differential dy when $dx = 0.01$ and $x = 2$, if $y = x^5 - 4x^3$. Explain what you've found.

Example: Find the differential dy when $dx = -0.2$ and $x = 1$, if $y = x^2 e^x$. Explain what you've found.

Example: Without a calculator, use differentials to approximate $\sqrt{4.2}$.

As we move from a point c to a nearby point $c + dx$, we can describe the change in f three ways:

	TRUE	**ESTIMATED**
Absolute change		
Relative change		
Percentage change		

Example: The range R of a projectile is $R = \frac{{v_0}^2}{32}(\sin 2\theta)$, where ${v_0}^2$ is the initial velocity in feet per second and θ is the angle of elevation. If $v_0 = 2200$ feet per second and θ changed from 10° to 11°, use differentials to approximate the change in the range.

Example: The measurement of a side of a square is found to be 15 centimeters. The possible error in measuring the side is 0.05 centimeter.

a) Approximate the percent error in computing the area of the square.

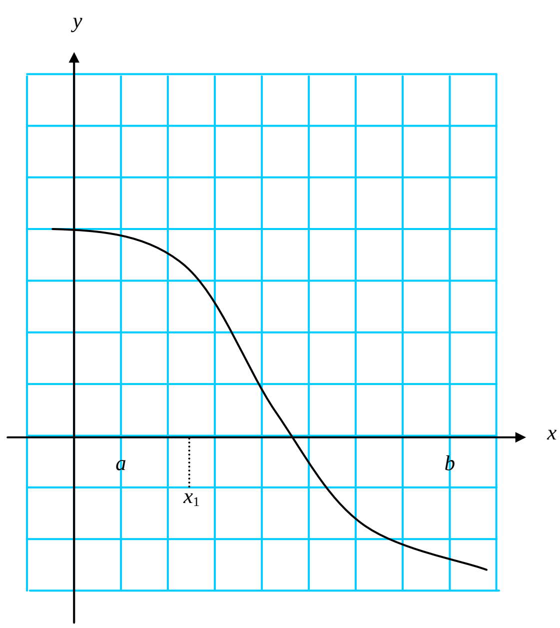
b) Estimate the maximum allowable percent error in measuring the side if the error in computing the area cannot exceed 2.5%.

Example: A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5°. How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

Newton's Method (no longer part of the AB curriculum) … time permitting …

Newton's method is a process using linearization to approximate (amazingly accurately) the zeroes of a function.

Example: Consider a continuous, differentiable function such that $f(a) > 0$ and $f(b) < 0$. What does the Intermediate Value theorem guarantee must occur between *a* and *b*?



Newton's method is based on the assumption that the tangent line at a point crosses the *x* axis at *about* the same place as the function. Since it is relatively easy to calculate the *x* – intercept of a line, we use this line to create a new estimate of the zero.

Example: Locate the function value at the point x_1. Draw a tangent line at that point.

Example: **Label** the *x* – intercept of the tangent line x_2. **Locate** the function value at x_2. **Draw** the tangent line at this point and **find** the *x* –intercept of the line.

Example: Is this point close to the point where the original function crosses the *x* – axis?

Example: Repeat the process one more time. How close do you get to the actual *x* – intercept?

Example: Can you think of a situation where this method would NOT work?

4.6 RELATED RATES

We have been taking derivatives with respect to x for almost the entire year. When even took derivatives of y implicitly, but as a function of x. Suppose a particle is moving along the curve $y = x^2$. As it moves, the x and y coordinates are changing at the same time. If we viewed their change as <u>a function of time</u>, then we could represent the rate of change in the y values with respect to time as $\frac{dy}{dt}$ and the rate of change in the x values with respect to time as $\frac{dx}{dt}$. We find these derivatives in the same way you found derivatives implicitly.

Example: Suppose both y and x are differentiable functions of t. Differentiate $y = x^2$ with respect to t.

Example: If you wanted to solve for $\frac{dy}{dt}$, what other information would have to be given to you?

Example: Suppose you are told that the particle moving along the curve $y = x^2$ is moving horizontally at 2 cm/s. Find the rate of change in the particle's vertical position at the exact moment the particle is at (3, 9).

In a related rates problem, you have an equation relating two or more things that <u>*change over time*</u>, and we want to find the rate of change (a derivative) of one of these things. It is important to understand that without some conditions given to use, we cannot solve the problem.

Guidelines for Solving Related-Rate Problems

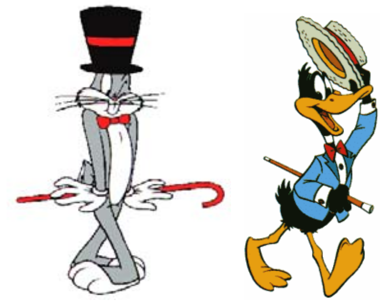
<u>Step 1</u>: Read the problem, really! You'd be amazed how many people skip this step. Then read it again! ☺
<u>Step 2</u>: Draw a diagram showing what's going on. Identify all relevant information and assign variables to what's changing. Use the general case (numbers for values that NEVER change in this situation, and variables for anything that is changing).

♬: Related Rates usually involve motion … any diagram you draw is like a still picture of what is occurring. Any part of your picture that NEVER changes can be labeled with a constant (or number), but any part of your picture that is in motion or is changing MUST be labeled with a variable!

In other words, if the radius of a circle is increasing and you are asked to find the rate of change in the area at the exact moment when the radius is 5 cm, then your diagram would be a circle, but you would NOT label the radius 5 because it is changing … you would label the radius r.

<u>Step 3</u>: Find the equation that gives the <u>relationship between the variables</u> you just named in step 2. This is sometimes the hardest part, but most problems fall into three categories … a triangle that you can use a trigonometric ratio (involving sides and angles), the Pythagorean theorem (involving all 3 sides of a right triangle), or a known formula like Area, Volume, Distance, etc.
<u>Step 4</u>: Find the particular information (values of variables at the exact moment you drew your diagram) for the problem and write it down, and list what you are looking for (normally this would be a derivative).
<u>Step 5</u>: Implicitly differentiate the equation with respect to time. Usually this equation will have at least two derivatives. If it has more than two, be sure you have enough information, or you may have find a relationship between two of the variables, and rewrite the equation in step 3 using this relationship.
<u>Step 6</u>: Plug in the particular information, and solve for the desired quantity. DO NOT DO THIS UNTIL AFTER YOU HAVE TAKEN THE DERIVATIVE!
<u>Step 7</u>: Write down your answer and circle it with your favorite color. (be sure to use correct units)

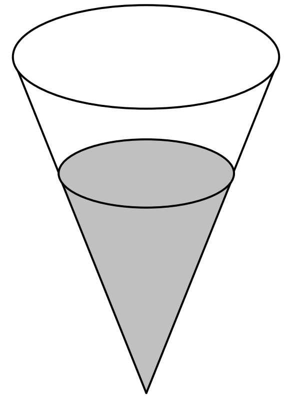
Example: Bugs and Daffy finished their final act on the *Bugs and Daffy Show* by dancing off the stage with a spotlight covering their every move. If they are moving off the stage along a straight path at a speed of 4 ft/s, and the spotlight is 20 ft away from this path, what rate is the spotlight rotating when they are 15 feet from the point on the path closest to the spotlight?



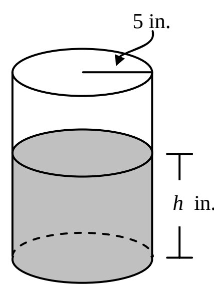
Example: Tweety is resting in a bird house 24 feet off the ground. Using a 26 foot ladder which he leaned against the pole holding the bird house, Sylvester tries to steal the small yellow bird. Tweety's bodyguard, Hector the dog, starts pulling the base of the ladder away from the pole at a rate of 2 ft/s. How fast is the ladder falling when it is 10 feet off the ground?

OK … I couldn't find a decent looney tunes picture for the next problem, so I thought I'd just throw in this cartoon (which by the way has nothing to do with related rates!) since I found it looking for any other good pictures. Besides, poor Wile E. Coyote has been working so much this year, it's about time he finally got a good meal. ☺

Example: A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep. The volume of a circular cone with radius *r* and height *h* is given by $V = \frac{1}{3}\pi r^2 h$.



Example: A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure to the right. Let *h* be the depth of the coffee in the pot, measured in inches, where *h* is a function of time *t*, measured in seconds. The volume *V* of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume *V* of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$.) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.



Example: A baseball diamond has the shape of a square with sides 90 feet long. Tweety is just flying around the bases, running from 2nd base (top of the diamond) to third base (left side of diamond) at a speed of 28 feet per second. When Tweety is 30 feet from third base, at what rate is Tweety's distance from home plate (bottom of diamond) changing?