5.1 ESTIMATING WITH FINITE SUMS

Example: Suppose from the 2nd to 4th hour of your road trip, you travel with the cruise control set to exactly 70 miles per hour for that two hour stretch. How far have you traveled during this time?

Example: Sketch a graph modeling the situation in the above example. Geometrically, how can we indicate the total distance traveled?

Example: What if the velocity was NOT constant. Say, for instance the velocity in miles per hour is given by the function $v(t)=10t-t^2$, where *t* is in hours, and we wanted to know the total distance traveled during the first 10 hours. Sketch this graph below. Geometrically speaking, do you think we can find the total distance traveled in the same was as before? Why or why not?

Word of caution to those brave few who are actually reading this … The following paragraphs are extremely important to the conceptual understanding of what we are about to do in Calculus. However, since you really haven't done anything yet, it might make you a little dizzy at first, so come back and read it again later. If you're reading this for the first time, you might experience that same feeling you get when you've been on the Tilt –A – Whirl one too many times at the fair. (Never been on the Tilt – A – Whirl? … well, take my word for it, it's not something you want to ride 10 minutes after eating a corn dog and a funnel cake!) Well, I warned you, but you've kept on reading anyway, so here it goes …

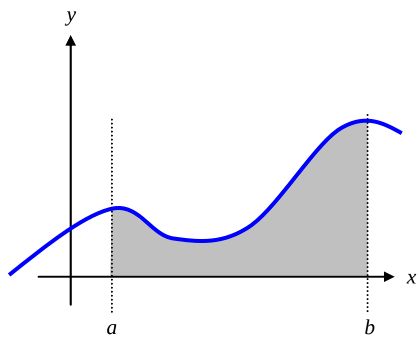
The key to finding the total distance traveled in the last example in a method similar to the first example is to break the time intervals into such short segments, that the velocity over those time segments is almost constant (this will require quite a few intervals). If the velocity is almost constant for each time interval, then we can find the distance traveled for each time interval (which is just the area of an extremely thin rectangle) and add all the areas of all the rectangles together. Sounds simple enough, right? Can you guess what extremely important calculus concept is involved?

We will spend MUCH more time with this later, but it turns out that if we are given the graph of a rate of change (like velocity in miles per hour) we will be able to find the total accumulated change over an interval (like total distance traveled, in mile) by finding the area under the curve.

OK, that last paragraph or two may not have made perfect sense to you … YET! … For now, THE GOAL is to figure out a way to find the area under the curve. This chapter actually discusses 5 ways to approximate this area, but we're only going to deal with 4 of them.

The Area Problem and the Rectangular Approximation Method (RAM)

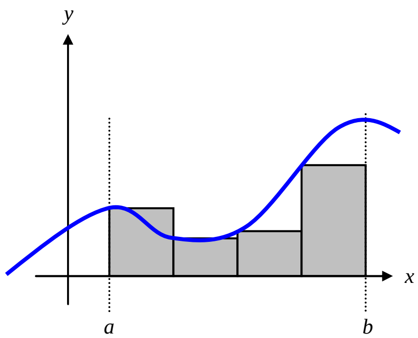
The limit process can be used to find the area under a curve, and we will get into this in more detail in the next section. Suppose we wanted to know the area of the region bounded by a curve, the x – axis, and the lines $x = a$ and $x = b$, as shown below.

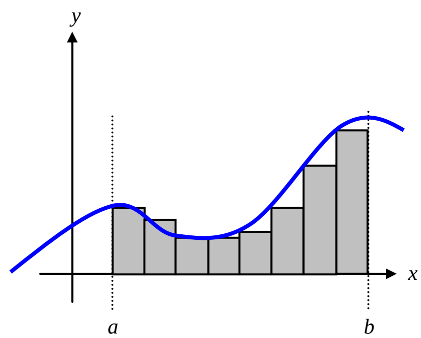


The first step is to divide the interval from a to b into subintervals. The examples below show 4 and 8 subintervals, respectively.

After dividing the given interval into subintervals, we can then draw rectangles using the width of each subinterval.

The height of each rectangle is determined by the function value at a point in the specific subinterval, and can be determined using 3 different methods. We could use the left endpoint of each subinterval (called LRAM), the right endpoint of each subinterval (RRAM), or the midpoint of each subinterval (MRAM). Which method is shown below?





Example: The total area under the curve then is approximately equal to the total area of all the rectangles. Which of the graphs above gives a better approximation of the area under the curve? Why? How could it be further improved?

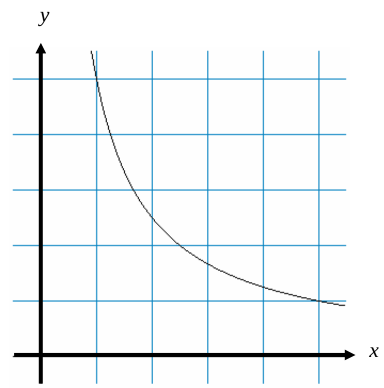
Summary of the Process: A sketch is almost mandatory!

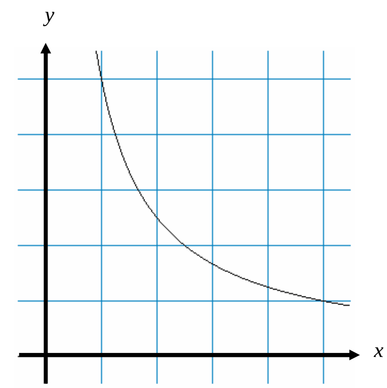
Step 1: Divide (or Partition) the interval into n subintervals.

Step 2: Create n rectangles whose base equals the width of each subinterval and whose height is determined by the function value at the left endpoint, the right endpoint, or the midpoint of the subinterval.

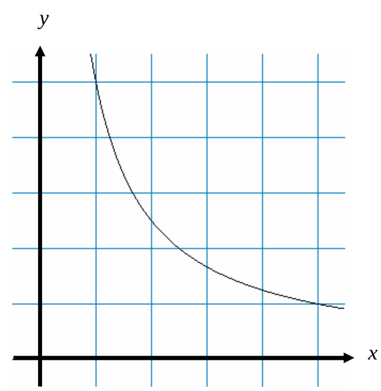
Step 3: Find the area of all n rectangles and add them together.

Example: The graph of $y = \dfrac{5}{x}$ is shown twice below. On the left picture approximate the area under the curve from $x = 1$ to $x = 5$ using LRAM with 4 rectangles. On the right picture, approximate the area under the curve from $x = 1$ to $x = 5$ using RRAM with 4 rectangles. Sketch the rectangles on each curve.





Example: Approximate the area under the curve from $x = 1$ to $x = 5$ using MRAM with 4 rectangles. Sketch the rectangles on the curve.



Example: It is not necessary to have a graph to estimate the area. Suppose the table below shows the velocity of a model train engine moving along a track for 10 seconds. Estimate the distance traveled by the engine, using 10 subintervals of length 1 with (*a*) left – endpoint values (LRAM) and (*b*) right – endpoint values (RRAM)

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

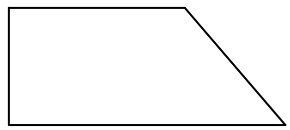
♬: All the examples on this page had a subinterval length equal to 1. This is not always the case, but was done to make the initial examples straightforward. Try doing the first example again using 5 rectangles instead of 4.

The Trapezoidal Rule (Really §5.5)

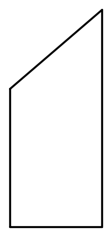
While rectangles make a fairly good approximation, it's easy to see that we're going to need a lot of them to provide a good estimate. We can find a better estimate in less time if we use trapezoids. If we were to partition the interval into subintervals like we did before, we can use each subinterval to create a trapezoid if we just connect the function values of the left and right endpoints. Before we begin, let's make sure you understand the area formula for a trapezoid.

$$\text{Area of a Trapezoid: } A = \frac{1}{2} \cdot h \cdot (b_1 + b_2)$$

While not all trapezoids must look like this, the one's we're going to be using will, so we'll stick with this picture. Label all the parts of the area formula on the picture below.

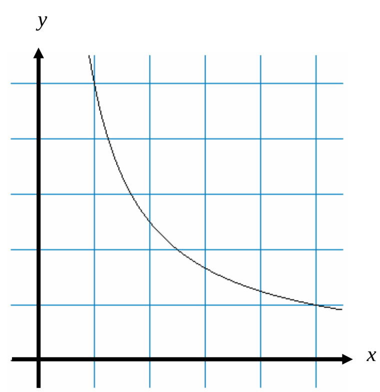


The biggest difference will be the orientation of the trapezoid. The ones we are going to be drawing will look like



Draw a set of axes on the picture above and a function that goes through the top left and top right points of the trapezoid. The "height" of the trapezoid is just the width of a subinterval, and the "bases" are going to be the function values of the left and right endpoints.

Example: Let's go back to the same function we used before. Use 4 trapezoids to approximate the area under the curve $y = \frac{5}{x}$ from $x = 1$ to $x = 5$. Sketch the trapezoids on the curve.



While all we're really doing is finding the area of a bunch of trapezoids, there is always a formula available.

The Trapezoid Rule

To approximate the area under a curve on the interval [a, b] use

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

where [a, b] is partitioned into n subintervals of equal length $h = \frac{(b-a)}{n}$.

Proof:

For all you formula memorizers … the key words in the above formula is EQUAL LENGTH. It doesn't work so well in the following example:

Example: [1998 AP Calculus AB #85 … with calculator] The function f is continuous on the closed interval [2, 8] and has values that are given in the table below.

x	2	5	7	8
$f(x)$	10	30	40	20

Using the subintervals [2, 5], [5, 7], and [7, 8], what is the trapezoidal approximation of the area under the curve?

A) 110 B) 130 C) 160 D) 190 E) 210

By the way … The trapezoid rule connects the left and right hand endpoints with a segment. This method of approximation turns out to be a pretty good, but if you were to connect the endpoints with a curve (namely a parabola) the approximation would be even better. Connecting the endpoints with a parabola and finding the area of the resulting shape is the basis behind the fifth method of approximation called Simpson's Rule. You can read about it on pages 291 – 293 if you find yourself just dying of curiousity, but it's not on the AP exam.

5.2 DEFINITE INTEGRALS

Riemann Sums

In the last section we found the area under a curve by finding the area of a finite number of rectangles (LRAM, RRAM, and MRAM) or a finite number of trapezoids (TRAPEZOID RULE). Every one of these was an example of what is called a **Riemann Sum**. We're going to stick with RECTANGLES for the time being.

The following steps illustrate what has to happen in order for the sum to be considered a RIEMANN Sum.
Read each of the following steps and then illustrate these on the function at the bottom of the page.

Step 1: Start with a *continuous* function on a *closed* interval.

Step 2: ***Partition*** the interval into n subintervals. (Previously we made sure they were all the same size, but it turns out, it really doesn't matter for what we're doing here.)

The kth subinterval will be $\left[x_{k-1}, x_k\right]$ and will have length $\triangle x_k = x_k - x_{k-1}$

Step 3: In each subinterval, pick ***any*** number and call the number picked from the kth subinterval c_k (for LRAM we picked the left endpoint … for RRAM we picked the right endpoint … for MRAM we picked the midpoint)

Step 4: For each interval, using the width, $\triangle x_k$, of the interval as the base, create a rectangle that extends from the x – axis to the function value, $f\left(c_k\right)$, of the number you picked in each interval.

♬: Some of these rectangles could lie below the x – axis.

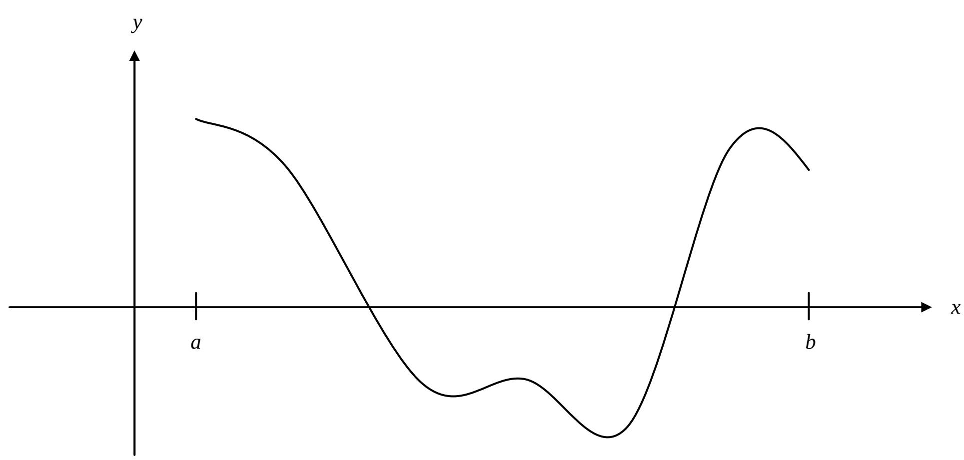
Step 5: On each interval, form the product $f\left(c_k\right)\cdot\triangle x_k$. (This would have been the area of the rectangles we found previously … but since any of the rectangles below the x – axis would have a negative product, we don't call this the area)

Step 6: Find the SUM of each of these products.

In other words find $f\left(c_1\right)\cdot\triangle x_1 + f\left(c_2\right)\cdot\triangle x_2 + f\left(c_3\right)\cdot\triangle x_3 + f\left(c_4\right)\cdot\triangle x_4 + \cdots + f\left(c_n\right)\cdot\triangle x_n$.

Another way to write this is $\sum_{k=1}^{n} f\left(c_k\right)\cdot\triangle x_k$

Following these steps gives you a **Riemann Sum for *f* on the interval [*a*, *b*]**. Every Riemann sum depends on the partition you choose (i.e. the number of subintervals) and your choice of the number within each interval, c_k.



Definite Integral as a Limit of a Riemann Sum

Goal: Develop a Mathematical DEFINITION of a Definite Integral … (i.e. What is a Definite Integral)

We used these sums to approximate the *area under the curve*. Again, since area is always positive, we cannot solely use this to describe all Riemann Sums. The same concept applies, however, when we want to find a better approximation.

Using the notation P to represent the partition, the longest subinterval length would be called the **norm** of P and is denoted $\|P\|$. One of the ways to ensure we have a better approximation is to increase the number of rectangles to infinity, and one of the ways to do this is to make sure that the norm of P goes to 0.

In order to make the lengths of all the subintervals infinitesimally small, we use a limit as follows:

Definition of the ***Definite Integral as a Limit of a Riemann Sum***

Let f be a function defined on a closed interval $[a, b]$. For any partition P of $[a, b]$ let the numbers c_k be chosen arbitrarily in the subintervals $[x_{k-1}, x_k]$. If there exists a number I such that

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k = I$$

no matter how P and the c_k's are chosen, then f is **INTEGRABLE** on $[a, b]$ and I is the **DEFINITE INTEGRAL** of f over $[a, b]$.

Important ♬ (Actually it's a Theorem): IF the function is continuous , THEN the Definite Integral will exist. However, the converse, while true some of the time is NOT ALWAYS true.

Another way to define Definite Integrals is $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot \Delta x$. Can you explain how this is different than the above definition and what it means for your partition?

Notation for Definite Integrals

The limit notation we used last is the form we will use to develop Integral notation. As the number of rectangles goes to infinity, the width of each rectangle, Δx, goes to zero. As we did in the section on differentials, we are going to use the notation dx to represent this infinitely tiny distance.

The summation notation of sigma is going to be replaced with an *Integral Sign* , $\int$, which look somewhat like a giant "S" for sum.

The $f(c_k)$ which represented a different function value for each interval is going to be replaced with $f(x)$ since the x – values are going to be soooooooo close together it's almost as if we are evaluating the function at EVERY x – value in the interval $[a, b]$. Combining all of this we have the following notation:

$$\int_a^b f(x)\,dx$$

We read the notation above as "***The Integral of f of x from a to b***"
Label each part of the notation above.

Using Definite Integrals as Area

Ok … back to the idea of Area. We can define the **area under the curve *y = f (x)* from *a* to *b*** as an *integral* from *a* to *b* AS LONG AS THE CURVE IS NONNEGATIVE AND INTEGRABLE on the closed interval [*a*, *b*].

Example: For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then evaluating each definite integral to find it.

a) $\int_{2}^{9} 3\, dx$

b) $\int_{-2}^{1} |x|\, dx$.

c) $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

Example: Consider the function $f(x) = 3 - x$. Sketch a graph of this function.

a) What is the "area" between the curve and the x – axis between $x = 4$ and $x = 8$?

b) Evaluate $\int_{4}^{8} (3 - x)\, dx$

c) Explain the difference between thinking of a definite integral as area under a curve in the first two examples and this example.

Example: If you knew that $\int_{0}^{\pi} \sin x\, dx = 2$, complete the following using what you know about definite integrals, area, and the sine function.

a) $\int_{\pi}^{2\pi} \sin x\, dx$

b) $\int_{0}^{2\pi} \sin x\, dx$

c) $\int_{0}^{\pi/2} \sin x\, dx$

d) $\int_{-\pi}^{\pi} \sin x\, dx$

e) $\int_{0}^{\pi} (2 + \sin x)\, dx$

f) $\int_{0}^{\pi} 2\sin x\, dx$

The FnInt Function of your TI – 83+

By this point, hopefully you understand the following concepts:

1. The limit of a Riemann Sum is used to define a Definite Integral
2. A Definite Integral can be used to find the Area under a curve if the curve is above the *x* – axis, and if the curve is below the *x* – axis the value of the definite integral is "negative area" … even though no one in their right mind would ever actually use that phrase in a math class if they wanted to be taken seriously!
3. Since the Definite Integral can be thought of as Area, you can draw a picture and use geometric formulas to find the areas.

BUT … what happens if you were asked right now, this instant, today to find the definite integral of a function that doesn't lend itself to nice geometric shapes?

The good news for now, is you don't even have to worry about how to do these by hand! You get to use your calculator!

How to Use Your Calculator to Find a Definite Integral:

The syntax for using your calculator is as follows: fnInt(*function*, *x*, *lower bound*, *upper bound*)

1. Press $\boxed{\text{MATH}}$
2. Press 9: fnInt(
3. Enter the function followed by $\boxed{,}$
4. Press $\boxed{\text{X,T,}\Theta,n}$ followed by $\boxed{,}$
5. Enter the lower bound followed by $\boxed{,}$
6. Enter the upper bound
7. Close the parenthesis

Example: Evaluate $\int_1^5 \frac{5}{x}\,dx$ … if you have your notes from the last section, go back and compare this answer to your answers for LRAM, RRAM, MRAM, and the Trapezoid Rule.

Example: Evaluate $3+2\int_0^{\pi/3} \tan x\,dx$

You can also do the same thing from the graphing screen.

Example: Graph $y=\sqrt{x}$ on a standard viewing window. Evaluate $\int_1^8 \sqrt{x}\,dx$.

Press $\boxed{\text{2nd}}\boxed{\text{TRACE}}$ (CALC), 7: ∫ f(x)dx, enter Lower Bound as 1, enter Upper Bound as 8.

Compare this to using to the previous method.

5.3 DEFINITE INTEGRALS AND ANTIDERIVATIVES

In the last section we defined the definite integral as a limit of a Riemann Sum, thus we can use the properties of limits to develop properties of the definite integral. The proofs of each of the rules below are derived directly from the properties of limits and Riemann Sums.

Rules for Definite Integrals

1. Order of Integration: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

   If you reverse the *order* of integration you get the opposite answer.

2. Zero: $\int_a^a f(x)\,dx = 0$

   This should make sense if you think about the "area" of a rectangle with no width.

3. Constant Multiple: If k is any constant, then $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$

   Taking the constant out of the integral many times makes it simpler to integrate.

4. Sum and Difference: $\int_a^b \left[f(x) \pm g(x) \right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

   This allows you to integrate functions that are added or subtracted separately. Notice, there are no rules here for two functions that are multiplied or divided … that comes later!

5. Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

   Pay close attention to the limits of integration … this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.

Example: Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, find the following:

a) $\int_2^6 \left[f(x) + g(x) \right] dx$ b) $\int_2^6 \left[g(x) - f(x) \right] dx$ c) $\int_2^6 3f(x)\,dx$

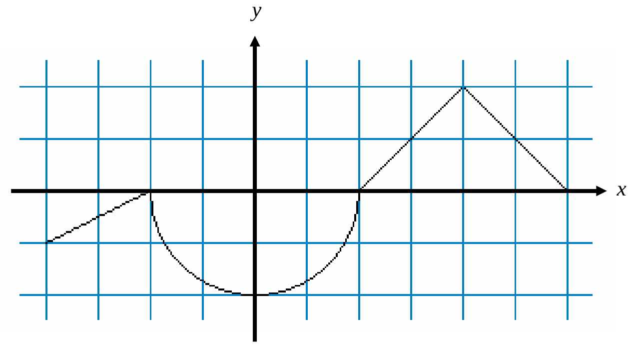
Example: Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, find the following:

a) $\int_0^7 f(x)\,dx$ b) $\int_5^0 f(x)\,dx$ c) $\int_5^5 f(x)\,dx$ d) $\int_0^5 3f(x)\,dx$

Example: The graph of f shown below consists of line segments and a semicircle. Evaluate each definite integral.

a) $\int_0^2 f(x)\,dx$ b) $\int_2^6 f(x)\,dx$ c) $\int_{-4}^2 f(x)\,dx$

d) $\int_{-4}^6 f(x)\,dx$ e) $\int_{-4}^6 |f(x)|\,dx$ f) $\int_{-4}^6 [f(x)+2]\,dx$



Example: Part *e* above, gives a way to find the total Area between the x – axis and the function between $x = –4$ and $x = 6$. Without using absolute value signs, write an expression that can be used to find the total area between the x – axis and the function between $x = –4$ and $x = 6$.

<u>*Average Value of a Function*</u>

Suppose you wanted to find the average temperature during a 24 hour period. How could you do it?

Suppose $f(t)$ represents the temperature at time t, measured in hours since midnight. One way to start is to measure the temperature at n equally spaced times $t_1, t_2, t_3, \ldots t_n$ and then average those temperatures.

Example: Using this method, write an expression for the AVERAGE temperature.

The larger the number of measurements, the more accurately this will reflect the average temperature. Notice we can write this expression as a Riemann sum by first noting that the interval between measurements will be $\Delta t = \frac{24}{n}$, so $n = \frac{24}{\Delta t}$.

Example: Substitute this value of n into your expression above and simplify.

Example: The last expression gives us an approximate Average Temperature. As $n \to \infty$ (meaning we are taking a lot of temperature readings) this Riemann Sum becomes a definite integral. Write the Definite Integral that gives us the Average Temperature since midnight.

Example: Do you think that there is any point during the day that the temperature reading on the thermometer is the *exact* value of the average temperature?
The process that we just used to find the average temperature is used to find the Average Value of any function.

The Average Value of a Function

If f is integrable on $[a, b]$, its **average value** on $[a, b]$ is given by

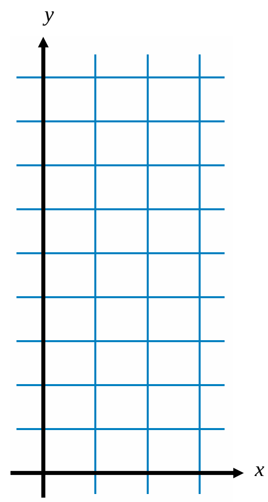
$$\text{AVERAGE VALUE} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example: What is the average value of $y = x^2\sqrt{x^3+1}$ on [0, 2] ?
[This question was asked on the 1997 AP Calculus AB exam (#27) and was intended to be done without a calculator … We have not yet discussed how this problem would be done without a calculator, so set up a definite integral and use your calculator to complete this problem]

A) $\frac{26}{9}$ B) $\frac{52}{9}$ C) $\frac{26}{3}$ D) $\frac{52}{3}$ E) 24

To get a more geometric idea of what the average value is, complete the following examples:

Example: Graph the function $y = x^2$ on [0,3] on the grid to the right.



Example: Set up a definite integral to find the average value of y on [0, 3], then use your calculator to evaluate the definite integral.

Example: Graph this as a value as a function on the grid to the right. Does the function every actually equal this value? If so, at what point(s) in the interval does the function assume its average value?

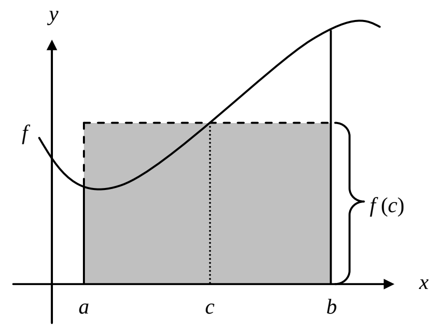
Example: What do you suppose is the relationship between the area between the x – axis and the curve $y = x^2$ on [0, 3] and the area of the rectangle formed using the average value as the height and the interval [0, 3] as the width?

The Mean Value Theorem for Definite Integrals

Do you remember the Mean Value Theorem we used for derivatives?

The Mean Value Theorem for Integrals basically says that if you are finding the area under a curve between $x = a$ and $x = b$, then there is *some* number c between a and b whose function value you can use to form a rectangle that has an area equal to the area under the curve.

Example: What is an expression that could be used to determine the *area under the curve* from a to b ?



Example: What is the area of the shaded rectangle?

The value of $f(c)$ is just the *Average Value* of f on the interval $[a, b]$. So another way to look at this is the Mean Value Theorem for Integrals just says that at some point within the interval the function MUST equal its average value.

> *Mean Value Theorem for Integrals*
>
> If f is continuous on $[a, b]$, then at some point c in $[a, b]$,
> $$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

It is greatly important that you understand the difference between *average rate of change* and *average value*.

Example: [Calculator Required] … Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

a) Is traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

b) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

c) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

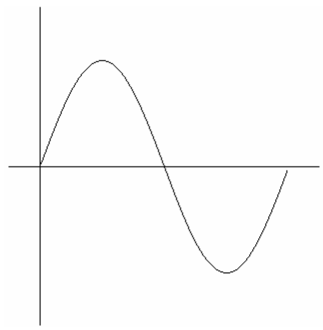
5.4 FUNDAMENTAL THEOREM OF CALCULUS

Do you remember the Fundamental Theorem of Algebra?

The Fundamental Theorem of Calculus has two parts. These two parts tie together the concept of integration and differentiation and is regarded by some to by the most important computational discovery in the history of mathematics!

In order to start developing this concept, we are going to use what I will call an "Area Accumulation Function".

Example: The graph of $f(t)$ given below has odd symmetry around the point (2, 0). On the interval [0, 2], the graph is symmetric with respect to the line $t = 1$. Also, $\int_0^1 f(t)\,dt = \frac{4}{3}$.



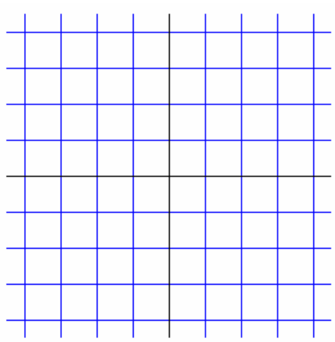
Example: Let $F(x) = \int_0^x f(t)\,dt$.

a) Complete the following table:

x	0	1	2	3	4
F (x)					

b) Sketch your best estimate of the graph of F on the grid below. (This is an "Area Accumulation Function")

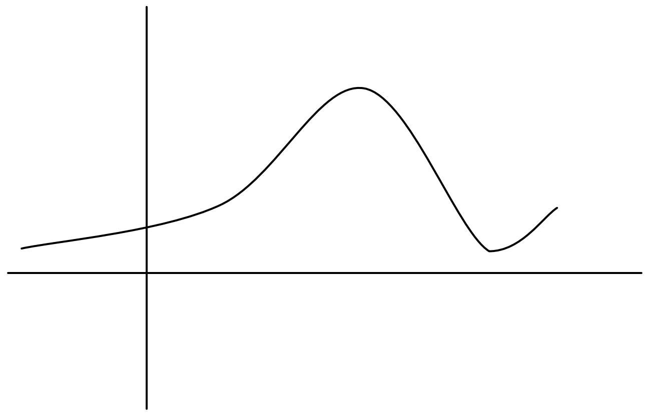
$$y = F(x)$$



Now that we've established what basically an "Area Accumulation Function" is, let's change the notation slightly just for this next part.

We're going to use ____________________ to describe the Area Accumulation Function from *a* to *x*.

Consider the graph below to be *f* (*t*). We are interested in the change in Area from $t = x$ to $t = x + h$.



Example: $A_a^x + A_x^{x+h} =$

Example: Solve the expression above for A_x^{x+h} :

Example: Does it really matter what value of *a* we use? The last expression can be written as ____________________.

In the last section, we talked about the properties of definite integrals. There was one we did not discuss, and we need it now.

#6. *Max-Min Inequality*: If max *f* and min *f* are the maximum and minimum values of *f* on [*a*, *b*], then

$$\min f \cdot (b-a) \leq \int_a^b f(x)\, dx \leq \max f \cdot (b-a)$$

Example: Using this property with our Area notation, we can bound A_x^{x+h} using

Example: Divide all three parts of the inequality above by *h*. What can be said then about $\dfrac{A_x^{x+h}}{h}$ as *h* goes to 0?

Hint: Remember the Sandwich (Squeeze) Theorem?

Example: Since the Sandwich Theorem tells us that $\lim_{h\to 0}\frac{A_x^{x+h}}{h} = f(x)$, replace A_x^{x+h} with the expression we used in a previous example.

Example: What is the left side of the equation above equal to?

What have we just done? We have said that the *derivative of the area accumulation function under* $f(x)$ is equal to $f(x)$. Instead of using the A notation to describe this, we could have used an integral.

The Fundamental Theorem of Calculus [Part #1 … Simple]

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

In other words, the Integral and the Derivative are just ___________________________.

Example: $\frac{d}{dx}\left[\int_3^x \left(5t^2 - 6t + 1\right) dt\right]$

Now, for part two!

Example: If we know that $A'(x) = f(x)$, then what does $A(x) =$

Example: How do we find C ?

Since $A(x)$ was are "area accumulation function" we can use an integral to represent $A(x)$ to obtain the second part of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus [Part 2: The Evaluation Part]

If f is continuous at every point of $[a, b]$,

$$\int_a^x f(x)\,dx = F(x) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

Example: $\int_0^3 x^2\,dx$

And now, for the last part … Part 1 Extended!

Example: Using the second part of the Fundamental Theorem of Calculus, $\int_a^x f(x)\,dx = F(x) - F(a)$, take the derivative of both sides. The result should be familiar.

What if *a* was not a constant and *x* was more complicated. In other words, what if the limits of integration were themselves functions of *x*?

Example: $\int_{v(x)}^{u(x)} f(t)\,dt =$

Example: Take the derivative of both sides of the equation above. Remember, $F'(x) = f(x)$.

We now have the extended version of the first part of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus [Part 1 … Extended]

$$\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)\,dt\right] = f(u(x))\cdot u'(x) - f(v(x))\cdot v'(x)$$

Example: