6.1 ANTIDERIVATIVES AND SLOPE FIELDS

Indefinite Integrals

In the previous chapter we dealt with definite integrals. Definite integrals had limits of integration . Indefinite integrals do not.

> The set of all antiderivatives of a function $f(x)$ in the **indefinite integral of f with respect to x** and is denoted
> $$\int f(x)\,dx$$

Recall, that all antiderivatives differ by a constant, so if $F'(x)=f(x)$, then $\int f(x)\,dx=F(x)+C$, where C is the constant of integration. The following table gives a list of results you should already be familiar with.

Integral Formulas

1. Power Rule for $n\neq -1$: $\int x^n\,dx=\dfrac{x^{n+1}}{n+1}+C$
2. Rule for $n = -1$: $\int \dfrac{1}{x}\,dx=\ln|x|+C$
3. $\int e^{kx}\,dx=\dfrac{e^{kx}}{k}+C$
4. $\int \sin(kx)\,dx=\dfrac{-\cos(kx)}{k}+C$
5. $\int \cos(kx)\,dx=\dfrac{\sin(kx)}{k}+C$
6. $\int \sec^2(x)\,dx=\tan(x)+C$
7. $\int \csc^2(x)\,dx=-\cot(x)+C$
8. $\int \sec(x)\tan(x)\,dx=\sec(x)+C$
9. $\int \csc(x)\cot(x)\,dx=-\csc(x)+C$

Example: Evaluate each integral.

a) $\int\left(-x^{-3}+\sqrt[3]{x}-1+e^{3x}\right)dx$

b) $\int(3\sin x-\sin 3x)\,dx$

Differential Equations

A **differential equation** is an equation containing a derivative. Just like in Algebra, when you want to solve an equation, you use an inverse operation. To "undo" a derivative we take an ____.

Recall, that a function can have many antiderivatives, all of which vary by a __________________________.

Solving a differential equation involves finding a unique equation that satisfies some *initial conditions* or *initial values*.

The **order** of a differential equation is the order of the highest derivative involved in the equation.

Example: Solve $\dfrac{dy}{dx}=\sin x$ by **separation of variables** if $y(0) = 2$.

A Graphical Look at Differential Equations

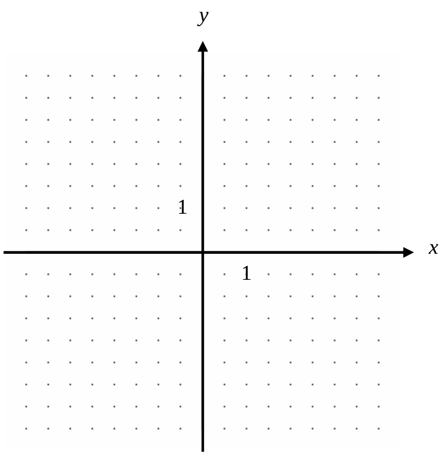
A **slope field** (or direction field) for the first order differential equation $\dfrac{dy}{dx} = f(x, y)$ is a plot of short line segments with slope $f(x, y)$ for a lattice of points (x, y) in the plane.

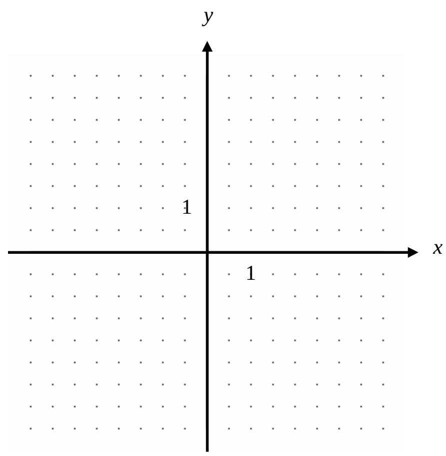
Example: On the diagram below, plot the slope field of the indicated differential equation.

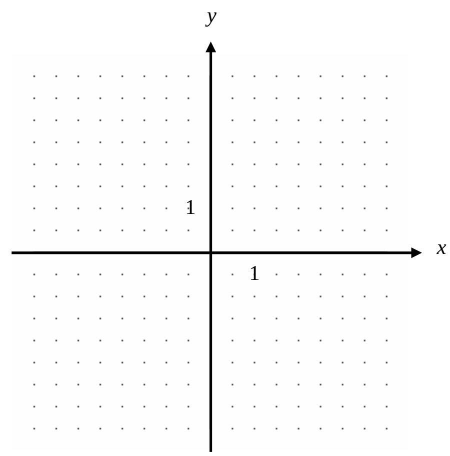
a) $\dfrac{dy}{dx} = x$

b) $\dfrac{dy}{dx} = y$

c) $\dfrac{dy}{dx} = -\dfrac{x}{y}$







Example: Suppose that you know that the point given below is on a particular solution of the differential equation given in the last example. By following slopes, draw on the diagrams above what you think the particular solution look like. (♬: The graph should follow the pattern of the slope field, but may go between the points rather than through them.)

a) (–2, –1)

b) (0, 0.5)

c) (2, 2)

Example: Solve the differential equations from the previous example by first **separating the variables**. Find the particular solution that contains the point given in the last example. Does your solution make sense when compared to the graph of the slope field?

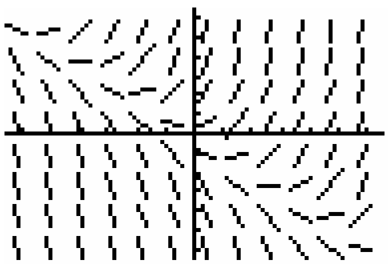
a) $\dfrac{dy}{dx} = x$

b) $\dfrac{dy}{dx} = y$

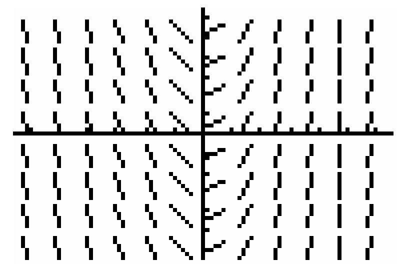
c) $\dfrac{dy}{dx} = -\dfrac{x}{y}$

Example: Solve the initial value problem $\frac{d^2y}{dx^2} = 2 - 6x$ given that $y(0) = 1$ and $y'(0) = 4$.

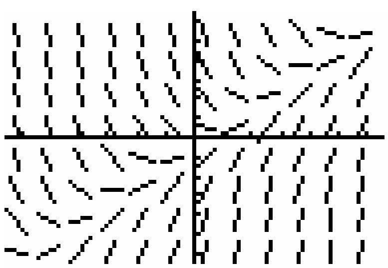
Example: Match the six slope fields shown below to their differential equations. Explain each choice.



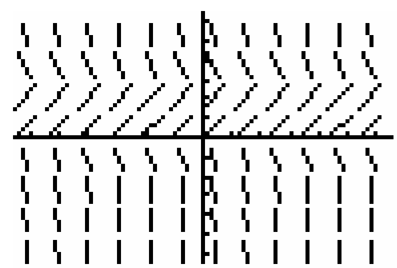
$\frac{dy}{dx} = x - y$



$\frac{dy}{dx} = 2x$

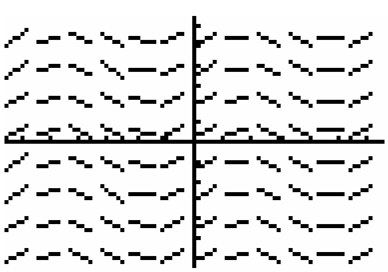


$\frac{dy}{dx} = 1 + y$

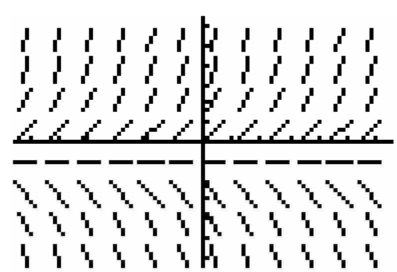


$\frac{dy}{dx} = \cos x$

$\frac{dy}{dx} = x + y$



$\frac{dy}{dx} = y(3 - y)$



6.2 INTEGRATION BY SUBSTITUTION

Up to this point, you have only been integrating functions by recognizing the function as an antiderivative of an elementary function. Integration by substitution allows us to integrate a function that was obtained by using the chain rule to take the derivative of a composite function.

The Guess-and-Check Method
Recall the way we differentiate composite functions using the Chain Rule:

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right)=f'\left(g\left(x\right)\right)\cdot g'\left(x\right).$$

Outside derivative Inside derivative

The Chain Rule gives us a product of two factors: the "outside derivative" and the "inside derivative." So, if a function has this form then it has an antiderivative.

Example: Use the Chain Rule to differentiate the following functions and state the outside and inside derivative.

a) $f(x)=\left(3x^4-5\right)^{12}$

b) $g(x)=\sin\left(x^3\right)$

Example. Use the pattern above to find the antiderivative of the following functions.

a) $\int\left(x^2-1\right)^3 2x\,dx$

b) $\int 3x^2\sqrt{x^3+2}\,dx$

c) $\int x\left(x^3-5\right)dx$

What problem did you find with the part (c)? How did you correct this problem?

Substitution Method

When the integrand is complicated, it helps to formalize this guess – and – check method as follows:

To Make a Substitution

Let u be the "inside function" and $du=u'(x)\,dx$

It is important to note, that with substitution, the goal is to substitute ALL values of the integrand with either *u* or *du*. Any "extras" must be accounted for, or substitution will NOT work!

Indefinite Integration with Substitution

Example: $\int x^3\sqrt{x^4+2}\,dx$

Example: $\int \sin^2(3x)\cos(3x)\,dx$

Example: $\int \tan x\,dx$

Example: $\int (x+1)\sqrt{2-x}\,dx$

Example: $\int \frac{x}{x^2+2}\,dx$

Example: $\int \frac{2x}{\sqrt{x^2+6}}\,dx$

Example: $\int \frac{e^x}{e^x+4}\,dx$

Example: $\int \frac{e^x+4}{e^x}\,dx$

Definite Integrals with Substitution

The only difference between indefinite integrals with substitution and definite integrals with substitution is the way in which you treat the limits of integration. Once you change variables, the limits no longer apply to the new variable.

Two methods: The first method involves using substitution to change the variables, then changing BACK into the original variable as before *before* you evaluate the definite integral. The second method involves using substitution to change the variable *and* changing the limits to correspond to that new variable. You then evaluate the definite integral using the converted limits.

♬: THE LIMITS MUST MATCH THE VARIABLE BEING USED, OR THERE MUST BE SOME NOTATION TO INDICATE THAT THE LIMITS BEING USED ARE DIFFERENT FROM THE VARIABLE BEING USED!

Method 1: Compute the indefinite integral, expressing the antiderivative in terms of the original variable, and then evaluate the result at the original limits.

Example: Compute $\int_0^1 x\sqrt{1-x^2}\,dx$.

Method 2: Convert the original limits to new limits in terms of the new variable and do not convert the antiderivative to back to the original variable.

Example: Evaluate $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$.

Example: Compute $\int_0^{\pi/4} \frac{\tan^3\theta}{\cos^2\theta}\,d\theta$.

Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Using these formulas involves "integrating by recognition". If you can recognize that the integral fits a particular format, then you know right away what the integral is.

Example: $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}}$

Example: $\displaystyle\int \frac{dx}{2 + 9x^2}$

Example: $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 9}}$

Sometimes, integrals do not quite fit any of these formulas. Other options include substitution and *completing the square*.

Example: $\displaystyle\int \frac{dx}{\sqrt{e^{2x} - 1}}$

Example: $\displaystyle\int \frac{x + 2}{\sqrt{4 - x^2}}\,dx$

Example: $\displaystyle\int \frac{dx}{x^2 - 4x + 7}$

6.3 INTEGRATION BY PARTS

When you integrate, you find an antiderivative, or you "undo" the derivative. When you used the substitution method, you were undoing a derivative that involved the Chain Rule. In this section we will learn how to undo a derivative that involved the Product Rule.

First, recall the way we differentiate functions using the Product Rule and differential notation.

$$(uv)' = u\ dv + v\ du$$

Using the Product Rule above, we will develop a "formula" for Integration by parts.

Begin by taking the Integral of both sides.

Then rearrange the equation, solving for $\int u\ dv$.

♬: You could have just as easily solved for $\int v\ du$, but it seems like every Calculus book uses the first way!

Integration by Parts

If *u* and *v* are functions of *x* and have continuous derivatives, then

$$\int u\ dv = uv - \int v\ du$$

Here's what you do … In order to solve integrals of this type, you need to decide what to let *u* and *dv* equal. Then follow the formula in the box above. **In order to find *v* from *dv* you must integrate, so choose wisely**.

Guidelines to Choose *u* and *dv*

- Let *dv* be the part that you are able to integrate.
- It helps if *du* is simpler than *u* (or at least no more complicated).
- It helps if *v* is simpler than *dv* (or at least no more complicated).

L I P E T

Further guidelines to choosing *u*:

First choice is a natural logarithm … *L* ….
Second choice is an inverse trigonometric function … *I* …
Third choice is a polynomial … *P* …
Fourth choice is an exponential … *E* …
Lastly, chose a trigonometric function … *T* …

Example: $\int xe^x \, dx$

Example: $\int \theta \sec\theta \tan\theta \, d\theta$

Example: $\int \frac{\ln x}{x^2} \, dx$

Example: $\int x^2 \ln x \, dx$

Special Cases of Integration by Parts

Example: $\int \arccos x \, dx$

Example: $\int \ln x \, dx$

Integrals of Trigonometric Functions

First, a few identities from trigonometry that you may or may not remember.

1. $1+\tan^2 x=\sec^2 x$ … everyone remembers this one, right?!

2. $\sin^2 x=\dfrac{1-\cos(2x)}{2}$ … This is called a Power Reducing Formula or the Half – angle identity for $\sin^2 x$, and if you remember this one, you probably don't need me to help you with the rest of this course ☺

3. $\cos^2 x=\dfrac{1+\cos(2x)}{2}$ … Same as #2 … It's a Power Reducing Formula or the Half – angle identity for $\cos^2 x$.

Can you integrate all of these functions? The first 4 should already be known.

$\int \sin x\,dx$ $\int \cos x\,dx$ $\int \sec^2 x\,dx$ $\int \csc^2 x\,dx$

$\int \tan x\,dx$ $\int \cot x\,dx$

$\int \csc x\,dx$ $\int \sec x\,dx$

$\int \sin^2 x\,dx$ $\int \cos^2 x\,dx$

$\int \tan^2 x\,dx$ $\int \cot^2 x\,dx$

Repeated Use of Integration by Parts

There are two ways to repeatedly use integration by parts. The first way just involves a repeated application of integration by parts until you reach an integral you can finish without integration by parts. The second involves using integration by parts and collecting like integrals on one side.

Example: $\int x^2 \cos x \, dx$

Example: $\int e^{2x} \sin x \, dx$

Tabular Method

In problems involving repeated applications of integration by parts, a tabular method can help to organize the work. Keep in mind, you are still using integration by parts repeatedly, but you are using the table to organize the repetitions. This method works well for integrals of the form $\int x^n \sin(ax)\, dx$, $\int x^n \cos(ax)\, dx$, $\int x^n e^{ax}\, dx$.

Example: $\int x^3 \sin(4x)\, dx$

Definite Integration with Integration by Parts

Example: $\int_0^{\pi/2} x \cos x \, dx$

Algebraic Techniques

Long Division

Example: $\int \frac{x^2-1}{x^2+1}\,dx$

Expand

Example: $\int (\sin x + \cos x)^2 \; dx$

Complete the Square:

Example: $\int \frac{2\,dx}{x^2-6x+10}$

Example: $\int_0^1 \frac{3\,dx}{(x+1)\sqrt{x^2+2x}}$

Example: $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

Separate the numerator

Example: $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$

Example: $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

6.4 EXPONENTIAL GROWTH AND DECAY

In many applications, the rate of change of a variable y is proportional to the value of y. If y is a function of time t, we can express this statement as

Example: Find the solution to this differential equation given the initial condition that $y = y_0$ when $t = 0$.

Exponential Growth and Decay Model

If y changes at a rate proportional to the amount present ($\frac{dy}{dt} = ky$) and $y = y_0$ when $t = 0$, then

$$y = y_0 e^{kt}$$

where k is the **proportional constant**.

Exponential **growth** occurs when $k > 0$, and exponential **decay** occurs when $k < 0$.

Example: The rate of change of y is proportional to y. When $t = 0$, $y = 2$. When $t = 2$, $y = 4$. What is the value of y when $t = 3$?

Example: [1985 AP Calculus BC #33] If $\frac{dy}{dt} = -2y$ and if $y = 1$ when $t = 0$, what is the value of t for which $y = \frac{1}{2}$?

A) $-\frac{1}{2}\ln 2$ B) $-\frac{1}{4}$ C) $\frac{1}{2}\ln 2$ D) $\frac{\sqrt{2}}{2}$ E) $\ln 2$

Example: **Radioactive Decay**: The rate at which a radioactive element decays (as measured by the number of nuclei that change per unit of time) is approximately proportional to the amount of nuclei present. Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram? [Pu-239 has a half life of 24,360 years]

Example: **Newton's Law of Cooling**: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature in the surrounding medium. A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3 °F. One hour later, the temperature of the body is 89.0 °F. The temperature of the room has been maintained at a constant 68 °F.

(a) Assuming the temperature, *T*, of the body obeys Newton's Law of Cooling, write a differential equation for *T*.

(b) Solve the differential equation to estimate the time the murder occurred.

Example: [1988 AP Calculus BC #43] Bacteria in a certain culture increase at rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

A) $\frac{3\ln 3}{\ln 2}$ B) $\frac{2\ln 3}{\ln 2}$ C) $\frac{\ln 3}{\ln 2}$ D) $\ln\left(\frac{27}{2}\right)$ E) $\ln\left(\frac{9}{2}\right)$

Example: [AP Calculus 1993 AB #42] A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

A) 4.2 pounds B) 4.6 pounds C) 4.8 pounds D) 5.6 pounds E) 6.5 pounds

Example: [1993 AP Calculus BC #38] During a certain epidemic, the number of people that are infected at any time increases at rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?

A) 343 B) 1,343 C) 1,367 D) 1,400 E) 2,057

Example: [1998 AP Calculus AB #84] Population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 10 years, then the value of k is

A) 0.069 B) 0.200 C) 0.301 D) 3.322 E) 5.000